

Riemannian Geometry Problem Sheets

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Problem Sheet 1

Various Structures

1.1 Distance and Length

Exercise 1.1.

- (1) Let (X, d) be a metric space. For any continuous curve $\gamma : [0, 1] \rightarrow X$, we define its length to be

$$L_d(\gamma) = \sup \left\{ \sum_{i=1}^n d(\gamma(t_{i-1}), \gamma(t_i)) \mid 0 = t_0 < t_1 < \dots < t_n = 1 \text{ is a subdivision of } [0, 1] \right\}$$

which could be $+\infty$. Then define a new “induced intrinsic metric” d_I on X via

$$d_I(x, y) := \inf \{ L_d(\gamma) \mid \gamma \text{ is a continuous curve joining } x \text{ and } y \}$$

which, again, could be $+\infty$.

- (a) Prove: $d(x, y) \leq d_I(x, y)$, and if $d_I(x, y) < +\infty$ for all x, y , then d_I is a metric on X .
- (b) A metric space (M, d) with $d = d_I$ is called a length space. Endow S^1 with the metric d inherited from the Euclidean distance. Is (S^1, d) a length space?
- (2) Let (M, g) be a connected Riemannian manifold, and d the Riemannian distance defined in chapter 3.
- (a) Prove: For any smooth curve $\gamma : [0, 1] \rightarrow M$, one has $\text{Length}_g(\gamma) = L_d(\gamma)$, where $\text{Length}_g(\gamma)$ is the length defined in chapter 3.
- (b) Prove: $d = d_I$.
- (3) Again let (M, g) be a connected Riemannian manifold. Let $\mathcal{C}$ be the set of all piecewise smooth curves $\gamma : [0, 1] \rightarrow M$, endowed with the uniform convergence topology. Prove: the “length functional”

$$\mathcal{L} : \mathcal{C} \rightarrow \mathbb{R}$$

$$\gamma \mapsto \text{Length}_g(\gamma)$$

is lower semi-continuous, i.e. if $\gamma_k \in \mathcal{C}$ and $\gamma_k \rightarrow \gamma$ uniformly, then

$$\liminf_{k \rightarrow \infty} \text{Length}_g(\gamma_k) \geq \text{Length}_g(\gamma).$$

1.2 Warped Products

Exercise 1.2. Let (M, g) , (N, h) be Riemannian manifolds, and ψ a positive smooth function on M . Define a warped product metric $g \times_\psi h$ on $M \times N$ via

$$(g \times_\psi h)((X_p, Y_q), (X'_p, Y'_q)) = g_p(X_p, X'_p) + \psi^2(p)h_q(Y_q, Y'_q).$$

- (1) Prove: $g \times_\psi h$ is a Riemannian metric.
- (2) Identify $\mathbb{R}^+ \times S^1$ with $\mathbb{R}^2 \setminus \{0\}$ via the polar coordinates, i.e.

$$\mathbb{R}^+ \times S^1 \rightarrow \mathbb{R}^2 \setminus \{0\},$$

$$(r, \theta) \mapsto (r \cos \theta, r \sin \theta).$$

Prove: The warped product metric on $\mathbb{R}^+ \times S^1$ with $\psi(r) = r$ coincides with the standard Euclidean metric on $\mathbb{R}^2 \setminus \{0\}$.

- (3) Identify $(0, \pi) \times S^{m-1}$ with $S^m - \{N, S\}$ (where N, S are the north/south poles of S^m respectively) via the map

$$(0, \pi) \times S^{m-1} \rightarrow S^m - \{N, S\} \subset \mathbb{R} \times \mathbb{R}^m, \\ (r, z) \mapsto (\cos r, (\sin r)z).$$

Prove: The warped product metric on $(0, \pi) \times S^{m-1}$ with $\psi(r) = \sin r$ coincides with the standard round metric on $S^m - \{N, S\}$.

1.3 Riemannian Covering

Exercise 1.3. Suppose M, N are connected smooth manifolds, and $\pi : M \rightarrow N$ is a smooth covering map.

- (1) Given any Riemannian metric g on N , we may endow with M the induced metric π^*g (called the covering metric) and call $\pi : (M, \pi^*g) \rightarrow (N, g)$ a Riemannian covering map. Prove: The covering metric π^*g is invariant under Deck transformations. [Recall that a Deck transformation is a diffeomorphism $\varphi : M \rightarrow M$ such that $\pi \circ \varphi = \pi$.]
- (2) Let $\pi : (M, \pi^*g) \rightarrow (N, g)$ be a Riemannian covering map, $\varphi : (P, h) \rightarrow (N, g)$ is a local isometry, and $\tilde{\varphi} : P \rightarrow M$ is a lift of φ . Prove: $\tilde{\varphi}$ is a local isometry.
- (3) Conversely, suppose $\pi : M \rightarrow N$ is a smooth normal covering map (i.e. the group of Deck transformations acts transitively on each fiber), g is a Riemannian metric that is invariant under all Deck transformations, then there exists a Riemannian metric π_*g on N such that $\pi : (M, \pi^*g) \rightarrow (N, g)$ is a Riemannian covering map.
- (4) Let $\vec{e}_1, \dots, \vec{e}_n$ be a basis of $\mathbb{R}^m$, and

$$\Gamma = \{k_1\vec{e}_1 + \dots + k_m\vec{e}_m \mid k_i \in \mathbb{Z}\}$$

be the lattice generated by these vectors. Starting with the Euclidean metric g_0 on $\mathbb{R}^m$, we may get an induced metric g_Γ on the torus $\mathbb{R}^m/\Gamma$ so that $\pi : (\mathbb{R}^m, g_0) \rightarrow (\mathbb{R}^m/\Gamma, g_\Gamma)$ is a Riemannian covering map. Prove: Two metrics g_Γ and $g_{\Gamma'}$ are isometric if and only if there exists an isometry of $(\mathbb{R}^m, g_0)$ that sends Γ to Γ' .

1.4 The Holonomy Group

Exercise 1.4. Let M be a connected smooth manifold with a connection ∇ . Consider the holonomy group

$$\text{Hol}_p(T_p M) = \{P_{0,1}^\gamma \mid \gamma : [0, 1] \rightarrow M \text{ is a piecewise smooth closed curve with } \gamma(0) = \gamma(1) = p\}$$

which is obviously a subgroup of $GL(T_p M)$. Prove:

- (1) For any $p \neq q$, $\text{Hol}_p(T_p M) = (P_{0,1}^\tau)^{-1} \text{Hol}_q(T_q M) P_{0,1}^\tau$, where τ is any piecewise smooth curve from $p = \tau(0)$ to $q = \tau(1)$.
- (2) If M is simply connected, then $\text{Hol}_p(T_p M)$ is connected.
- (3) If (M, g) is a Riemannian manifold and ∇ metric compatible, then $\text{Hol}_p(T_p M) \subset O(T_p M)$.
- (4) Find the holonomy groups of the standard $\mathbb{R}^n$, S^m and $\mathbb{H}^m$.
- (5) Find the relation between the holonomy groups of $(M_1 \times M_2, g_1 \times g_2)$ and (M_i, g_i) .

1.5 More Riemannian Metrics

Exercise 1.5. (1) An immersion $f : N \rightarrow \mathbb{R}^{m+1}$ of an m dimensional smooth manifold N into $\mathbb{R}^{m+1}$ is called a hypersurface. Suppose $\{U, u^1, \dots, u^m\}$ is a local chart on U so that the map f can be expressed locally as

$$x^k = f^k(u^1, \dots, u^m), \quad 1 \leq k \leq m+1,$$

where $(x^1, \dots, x^{m+1})$ are the coordinates in $\mathbb{R}^{m+1}$. Prove:

$$f^*g_0 \Big|_U = \sum_{k,i,j} \frac{\partial f^k}{\partial u^i} \frac{\partial f^k}{\partial u^j} du^i \otimes du^j.$$

- (2) A surface of revolution S in $\mathbb{R}^3$ can be formed by rotating a curve

$$\gamma(t) = (0, y(t), z(t)), \quad (a < t < b)$$

in the yz plane with respect to z -axis, where we assume

$$y(t) > 0 \text{ and } (y'(t))^2 + (z'(t))^2 \neq 0$$

for all t . As a consequence, we can parametrize the surface as

$$S(t, \theta) = (y(t) \cos \theta, y(t) \sin \theta, z(t)), \quad (a < t < b, 0 < \theta < 2\pi).$$

Write down the induced metric of g_0 in $\mathbb{R}^3$ on S .

1.6 Riemannian Geometry of S^2 , Part I

Exercise 1.6. Consider S^2 as a submanifold in $\mathbb{R}^3$. Using cylindrical coordinates θ and z we have seen

$$g_{S^2} = \frac{1}{1-z^2} dz \otimes dz + (1-z^2) d\theta \otimes d\theta.$$

- (1) Calculate the length of a great circle on S^2 .
- (2) For any two points, calculate the distance between them. [Hint: By rotating you can always assume one point is at the North pole, while the other point has $\theta = \pi$.]
- (3) Calculate the volume of $K_{a,b} = \{(z, \theta) \mid a < z < b\}$, where $-1 < a < b < 1$.
- (4) For any smooth vector field $X = X^\theta \partial_\theta + X^z \partial_z$, write down $\operatorname{div}(X)$.
- (5) For any smooth function $f = f(z, \theta)$, write down ∇f and Δf .

1.7 Different Models of Hyperbolic Spaces

Exercise 1.7.

- (1) (Hyperboloid Model) We equip with $\mathbb{R}^{m+1}$ the Lorentz inner product

$$\langle x, y \rangle := -x^0 y^0 + x^1 y^1 + \cdots + x^m y^m,$$

and let $H^m = \{x \in \mathbb{R}^{m+1} \mid \langle x, x \rangle = -1, x^0 > 0\}$. Show that $\langle \cdot, \cdot \rangle$ induces a Riemannian metric $g_{H^m} = \iota^* \langle \cdot, \cdot \rangle$ on H^m .

- (2) (Poincaré Disk Model) Let B^m be the open ball of radius 1 in $\mathbb{R}^m$, equipped with a Riemannian metric

$$g_{B^m} = \frac{4}{(1-|x|^2)^2} (dx^1 \otimes dx^1 + \cdots + dx^m \otimes dx^m).$$

Define a map

$$\begin{aligned} f : H^m &\rightarrow B^m, \\ (x^0, x^1, \dots, x^m) &\mapsto \frac{1}{1+x^0} (x^1, \dots, x^m). \end{aligned}$$

Prove: f is an isometry.

- (3) (Poincaré Half-plane Model) Let U^m be the upper half-space in $\mathbb{R}^m$ defined by $x^1 > 0$, equipped with a Riemannian metric

$$g_{U^m} = \frac{1}{(x^1)^2} (dx^1 \otimes dx^1 + \cdots + dx^m \otimes dx^m).$$

Define a map

$$\begin{aligned} g : B^m &\rightarrow U^m, \\ (x^1, \dots, x^m) &\mapsto \frac{(1 - (x^1)^2 - \cdots - (x^m)^2, 2x^2, \dots, 2x^m)}{(x^1 - 1)^2 + (x^2)^2 + \cdots + (x^m)^2}. \end{aligned}$$

Prove: g is an isometry.

1.8 The Laplace-Beltrami Operator

Exercise 1.8. Let (M, g) be a Riemannian manifold.

- (1) Let f, h be smooth functions on (M, g) . Check:

$$\Delta(fh) = f\Delta h - 2g(\nabla f, \nabla h) + h\Delta f.$$

- (2) Let $\bar{g} = e^{2\varphi}g$ be a Riemannian metric on M that is conformal to g . Prove:

$$\bar{\Delta}f = e^{-2\varphi}[\Delta f - (m-2)g(\nabla\varphi, \nabla f)],$$

where $\bar{\Delta}$ denote the Laplace-Beltrami operator associated to $\bar{g}$.

- (3) Find the relation between $\Delta_{\mathbb{R}^m}$ and $\Delta_{S^{m-1}}$.
 (4) Find the relation between $\Delta_{M \times N}$ and Δ_M, Δ_N .

1.9 Basic Spectral Geometry

Exercise 1.9. We say that λ is an eigenvalue of Δ if there exist smooth function $\varphi \neq 0$ such that

$$\Delta\varphi = \lambda\varphi.$$

The function φ is called an eigenfunction of Δ corresponding to λ . Now suppose (M, g) is a connected compact Riemannian manifold. Prove:

- (1) All eigenvalues of Δ are non-negative real numbers.
 (2) $\lambda = 0$ is always an eigenvalue, whose eigenfunctions are constants.
 (3) If φ is an eigenfunction of $\lambda > 0$, then $\int_M \varphi d\text{Vol} = 0$.
 (4) If φ_1 and φ_2 are eigenfunctions of different eigenvalues, then $\int_M \varphi_1 \varphi_2 d\text{Vol} = 0$.
 (5) Equip $S^1 \times S^1$ with the standard metric. Find all its eigenvalues and corresponding eigenfunctions.

1.10 Riemannian Manifold with Boundary

Exercise 1.10. In Chapter 4 we briefly discussed Riemannian manifold with boundary.

- (1) Prove divergence theorem II.
 (2) Prove Green's formula II.
 (3) We call $\lambda \in \mathbb{R}$ a Dirichlet eigenvalue of Δ if there exist smooth function $\varphi \neq 0$ such that

$$\Delta\varphi = \lambda\varphi \text{ in } M \quad \text{and} \quad \varphi = 0 \text{ on } \partial M.$$

- (a) Prove: All Dirichlet eigenvalues of Δ are positive real numbers.
 (b) Prove: If φ_1 and φ_2 are eigenfunctions of different eigenvalues, then $\int_M \varphi_1 \varphi_2 d\text{Vol} = 0$.

1.11 Killing Vector Fields

Exercise 1.11. Any vector field X defines a local family of diffeomorphisms

$$\begin{aligned} \phi_t^X : U \subset M &\rightarrow \phi(U) \subset M, \\ p &\mapsto \phi_t^X(p) = \gamma_{p, X_p}(t) \end{aligned}$$

for $-\varepsilon < t < \varepsilon$, which satisfies $\phi_t^X \circ \phi_s^X = \phi_{t+s}^X$ for $t, s, t+s \in (-\varepsilon, \varepsilon)$. Now suppose (M, g) is a Riemannian manifold. We say X is a Killing vector field if these ϕ_t^X 's are isometries.

- (1) Let $X, Y, Z \in \Gamma^\infty(TM)$. Prove: The following statements are equivalent:
 - (a) X is a Killing vector field.
 - (b) $\mathcal{L}_X g = 0$.
 - (c) $X\langle Y, Z \rangle = \langle [X, Y], Z \rangle + \langle Y, [X, Z] \rangle$.
 - (d) $\langle \nabla_Y X, Z \rangle + \langle \nabla_Z X, Y \rangle = 0$.
 - (e) The $(1,1)$ -tensor field ∇X defined by $\nabla X : Y \mapsto \nabla_Y X$ is skew-symmetric.
- (2) Let X be a Killing vector field, and $p \in M$ is a given point. Prove: X is uniquely determined by $X(p)$ and $\nabla X(p)$.
- (3) Prove: The set of Killing vector fields, denoted by $\mathfrak{iso}(M, g)$, is a Lie subalgebra of $(\Gamma^\infty(M), [\cdot, \cdot])$.
- (4) Prove: $\dim \mathfrak{iso}(M, g) \leq \frac{(m+1)m}{2}$, where $m = \dim M$.
- (5) Now suppose M is compact, in which case the isometry group $\text{Iso}(M, g)$ is known to be a Lie group. Assume this. Prove: $\mathfrak{iso}(M, g)$ is the Lie algebra of $\text{Iso}(M, g)$.

1.12 Bi-invariant Metric on Lie Groups

Exercise 1.12. A Riemannian metric on a Lie group G is left-invariant if

$$\langle X_g, Y_g \rangle_g = \langle (dL_h)_g X_g, (dL_h)_g Y_g \rangle_{hg}$$

holds for all $g, h \in G$ and all $X_g, Y_g \in T_g G$.

- (1) Take an arbitrary inner product $\langle \cdot, \cdot \rangle_e$ on $T_e G$. Check:

$$\langle X_g, Y_g \rangle := \langle (dL_{g^{-1}})_g X_g, (dL_{g^{-1}})_g Y_g \rangle_e$$

defines a left-invariant Riemannian metric on G .

- (2) Write down the definition for a Riemannian metric to be right-invariant, and construct a right-invariant Riemannian metric on G .
- (3) A Riemannian metric on a Lie group G is bi-invariant if it is both left-invariant and right-invariant. Prove: If $\langle \cdot, \cdot \rangle$ is a bi-invariant Riemannian metric on G , then $\langle \cdot, \cdot \rangle_e$ is a conjugation-invariant inner product on $T_e G$.
- (4) Conversely, show that if $\langle \cdot, \cdot \rangle_e$ is a conjugation-invariant inner product on $T_e G$, then the left-invariant Riemannian metric defined in (1) is bi-invariant.
- (5) Now suppose G is compact, so that there exists a Haar measure on G , which can be think of as a volume form ω that is both left-invariant and right-invariant:

$$L_g^* \omega = \omega = R_g^* \omega, \quad \forall g \in G.$$

Prove: For any left-invariant metric on G , the new inner product

$$\langle X_g, Y_g \rangle_g^{\text{new}} = \int_G \langle (dR_h)_g X_g, (dR_h)_g Y_g \rangle_{gh}^{\text{old}} \omega, \quad \forall g \in G, \text{ and } X_g, Y_g \in T_g G$$

defines a bi-invariant Riemannian metric on G .

Problem Sheet 2

Curvatures

2.1 Isometry Preserves Geometry

Exercise 2.1. Let $\varphi : (M, g) \rightarrow (\tilde{M}, \tilde{g})$ be a local isometry. Prove:

- (1) Given a coordinate system $(x^1, \dots, x^m)$ near a point p of M , one defines a coordinate system $(\tilde{x}^1, \dots, \tilde{x}^m)$ near $\varphi(p)$ by letting $\tilde{x}^i = x^i \circ \varphi^{-1}$. Prove: $\varphi^* \tilde{g}_{ij} = g_{ij}$.
- (2) Prove: For vector fields $X, Y \in \Gamma^\infty(TM)$, by restricting to neighborhoods of p and $\varphi(p)$, one has $d\varphi(\nabla_X Y) = \tilde{\nabla}_{d\varphi(X)} d\varphi(Y)$.
- (3) State and prove the fact “curvature tensor is invariant under local isometry”.
- (4) State and prove the fact “sectional curvature is invariant under local isometry”.
- (5) Prove that the natural action of the isometry group $\text{Iso}(S^m, g_{\text{round}})$ on the Grassmannian bundle $G_2(TS^m)$ is transitive, and thus (S^m, g_{round}) has constant sectional curvature.

2.2 Riemannian Geometry of the Hyperbolic Space: Part II

Exercise 2.2. Let H^m be the upper half-space in the $\mathbb{R}^m$, i.e.

$$H^m = \{(x^1, \dots, x^m) \in \mathbb{R}^m \mid x^m > 0\},$$

equipped with the hyperbolic metric

$$g_{H^m} = \frac{1}{(x^m)^2} (dx^1 \otimes dx^1 + \dots + dx^m \otimes dx^m).$$

- (1) Calculate the Christoffel's symbols.
- (2) Calculate the Riemannian curvature tensor, and show that H^m has constant sectional curvature -1 .
- (3) For $m = 2$ and $\varepsilon > 0$, find the volume of the vertical stripe

$$S_\varepsilon = \{(x, y) \mid |x| < 1, \varepsilon < y < \infty\}.$$

2.3 Riemannian Geometry of Product Manifolds

Exercise 2.3.

- Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds. Consider the product Riemannian manifold $(M_1 \times M_2, \pi_1^* g_1 + \pi_2^* g_2)$, where

$$\pi_i : M_1 \times M_2 \rightarrow M_i$$

be the canonical projections. For any $(p, q) \in M_1 \times M_2$, we let

$$\begin{aligned} \iota_1^q : M_1 &\rightarrow M_1 \times M_2, \\ p &\mapsto (p, q) \end{aligned}$$

be the embedding of M_1 into $M_1 \times M_2$ as $M_1 \times \{q\}$. Similarly one define $\iota_2^p : M_2 \rightarrow M_1 \times M_2$. Denote by ∇^i and Rm^i the Levi-Civita connection and Riemann curvature tensor on M_i .

- (1) Prove: The Levi-Civita connection ∇ on $M_1 \times M_2$ satisfies

$$\nabla_{X_1+X_2}(Y_1+Y_2)(p, q) = d\iota_1^q(\nabla_{X_1}^1 Y_1)_p + d\iota_2^p(\nabla_{X_2}^2 Y_2)_q$$

for any $X_1, Y_1 \in \Gamma^\infty(TM_1)$, $X_2, Y_2 \in \Gamma^\infty(TM_2)$.

- (2) Prove: The Riemann curvature tensor of $M_1 \times M_2$ is

$$Rm(X_1 + X_2, Y_1 + Y_2, Z_1 + Z_2, W_1 + W_2) = Rm^1(X_1, Y_1, Z_1, W_1) + Rm^2(X_2, Y_2, Z_2, W_2)$$

for any $X_1, Y_1, Z_1, W_1 \in \Gamma^\infty(TM_1)$, $X_2, Y_2, Z_2, W_2 \in \Gamma^\infty(TM_2)$.

- (3) Write down similar formula for the Ricci tensor, the Weyl tensor and the scalar curvature of $M_1 \times M_2$ in terms of those of M_1 and M_2 .

- Now consider $S^m \times S^n$ endowed with the canonical metric.

- (4) Prove: All sectional curvatures of $S^m \times S^n$ lie in $[0, 1]$.

- (5) Find the Ricci and Weyl tensor of $S^m \times S^n$.

- (6) When will it be Einstein? When will it be locally conformally flat?

- (7) Find the scalar curvature of $S^m \times S^n$.

2.4 Polarization Formula for Riemann Curvature Tensor

Exercise 2.4. Denote $\tilde{K}(X, Y) = Rm(x, Y, X, Y)$.

- (1) Find a formula for $Rm(X, Y, X, W)$ in terms of $\tilde{K}$.

- (2) Prove:

$$\begin{aligned} 6Rm(X, Y, Z, W) &= \tilde{K}(X + Z, Y + W) - \tilde{K}(Y + Z, X + W) - \tilde{K}(X, Y + W) - \tilde{K}(Z, Y + W) \\ &\quad - \tilde{K}(X + Z, Y) - \tilde{K}(X + Z, W) + \tilde{K}(Y, X + W) + \tilde{K}(Z, X + W) \\ &\quad + \tilde{K}(Y + Z, X) + \tilde{K}(Y + Z, W) + \tilde{K}(X, W) + \tilde{K}(Z, Y) - \tilde{K}(Y, W) - \tilde{K}(Z, X). \end{aligned}$$

2.5 Curvature-like Tensors as Curvature Tensors

Exercise 2.5.

- (1) Fix r . For $1 \leq i \leq j \leq m$, choose vectors $h_{ij} \in \mathbb{R}^r$, and let $h_{ji} = h_{ij}$. Define

$$R_{ijkl} = \langle h_{ik}, h_{jl} \rangle - \langle h_{il}, h_{jk} \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^r$. Prove: $R_{ijkl}e^i \otimes e^j \otimes e^k \otimes e^l$ is a curvature-like tensor, where $e^1, \dots, e^m$ is a basis of $(\mathbb{R}^m)^*$.

- (2) Prove: Fix m , for r large enough, any curvature-like tensor on $\mathbb{R}^m$ arises in this way.

2.6 Riemannian Geometry under Conformal Change

Exercise 2.6. Let (M, g) be a Riemannian manifold of dimension m . Let $\bar{g} = e^{2\varphi}g$ be a Riemannian metric that is conformal to g . In what follows, everything without a “bar” (e.g. ∇ , Rm , S , etc) is for (M, g) , and everything with a “bar” (e.g. $\bar{\nabla}$, $\bar{Rm}$, $\bar{S}$, etc) is for $(M, \bar{g})$.

- (1) Prove: $\bar{\nabla}_X Y = \nabla_X Y + (X\varphi)Y + (Y\varphi)X - g(X, Y)\nabla\varphi$.

- (2) Prove: $\bar{Rm} = e^{2\varphi}[Rm - g \otimes (\nabla d\varphi - d\varphi \otimes d\varphi + \frac{1}{2}|d\varphi|^2 g)]$.

- (3) Prove: $\bar{S} = e^{-2\varphi}[S + 2(m-1)\Delta\varphi - (m-2)(m-1)|\nabla\varphi|^2]$.

- (4) The conformal Laplacian is defined to be the operator

$$Lf = \Delta f + \frac{(m-2)S}{4(m-1)}f.$$

Check: For any $f \in C^\infty(M)$,

$$\bar{L}f = e^{-\frac{m+2}{2}\varphi} L(e^{\frac{m-2}{2}\varphi} f).$$

- (5) Deduce the Yamabe equation: If $m > 2$, and $\bar{g} = u^{\frac{4}{m-2}}g$, then

$$4\frac{m-1}{m-2}\Delta u + S \cdot u = \bar{S} \cdot u^{\frac{m+2}{m-2}}.$$

2.7 Riemannian Geometry of Surfaces

Exercise 2.7. Let M be a two dimensional manifold, i.e. a surface, with Riemannian metric g .

- (1) Prove: Near each point p , there is a coordinate system so that g is of the form

$$g = E(x^1, x^2)dx^1 \otimes dx^1 + G(x^1, x^2)dx^1 \otimes dx^2.$$

- (2) Express the sectional curvature of M at each point using the functions E, G above. [You can use either direct computation or the moving frame.]

2.8 Weyl Tensors

Exercise 2.8.

- (1) Let (M, g) be of constant sectional curvature. Prove: $\mathbb{R} \times M$ is locally conformally flat.
- (2) Again suppose (M, g) has constant sectional curvature. Suppose ψ is a positive smooth function on $\mathbb{R}$. Prove: The warped product $\mathbb{R} \times_{\psi} M$ is locally conformally flat.
- (3) Find the Weyl tensor of $S^2 \times S^2$ and $\mathbb{R}^2 \times S^2$.

2.9 Riemannian Geometry of Submanifolds: Part I

Exercise 2.9. Let $(\bar{M}, g_{\bar{M}})$ be a Riemannian manifold, and $\iota : M \hookrightarrow \bar{M}$ be an embedded submanifold, with induced Riemannian structure $g_M = \iota^*g_{\bar{M}}$. Using the Riemannian structure of $\bar{M}$, for each point $p \in M \subset \bar{M}$, any $\bar{X}_p \in T_p\bar{M}$ can be written uniquely as

$$\bar{X}_p = \bar{X}_p^T + \bar{X}_p^{\perp},$$

where $\bar{X}_p^T \in T_pM$ and $\bar{X}_p^{\perp} \in (T_pM)^{\perp}$. In what follows we assume $X, Y, Z, W \in \Gamma(TM)$ are smooth tangent vector fields on M , and $\xi, \zeta \in \Gamma(TM^{\perp})$ are smooth normal vector fields on M , i.e. $\xi_p, \zeta_p \in (T_pM)^{\perp}$ for all $p \in M$ and are smooth.

- (1) Let $\bar{\nabla}$ be the Levi-Civita connection of $(\bar{M}, g_{\bar{M}})$. Explain: $\bar{\nabla}_X Y$ is well-defined.
- (2) Let $\nabla_X Y = (\bar{\nabla}_X Y)^T$. Check: ∇ is the Levi-Civita connection on M .
- (3) Set $S(X, Y) = (\bar{\nabla}_X Y)^T$. Check: S is $C^{\infty}(M)$ -linear and symmetric. (S is called the second fundamental form of M .)
- (4) Define the shape operator $S_{\xi} : T_pM \rightarrow T_pM$ by

$$\langle S_{\xi}(X), Y \rangle = \langle S(X, Y), \xi \rangle, \quad \forall Y \in \Gamma^{\infty}(TM).$$

Prove: S_{ξ} is symmetric.

- (5) Prove: $S_{\xi}(X) = -(\bar{\nabla}_X \xi)^T$.
- (6) Let $\bar{Rm}, Rm$ be the Riemannian curvature tensor on $\bar{M}, M$. Prove: (Gauss equation)

$$\bar{Rm}(X, Y, Z, W) = Rm(X, Y, Z, W) - \langle S(X, Z), S(Y, W) \rangle + \langle S(X, W), S(Y, Z) \rangle.$$

- (7) Use the Gauss equation to prove: (S^m, g_{S^m}) has constant curvature 1.
- (8) Verify the Codazzi equation

$$\bar{Rm}(X, Y, Z, \xi) = -\langle (\bar{\nabla}_X S)(Y, Z), \xi \rangle + \langle (\bar{\nabla}_Y S)(X, Z), \xi \rangle,$$

where $(\bar{\nabla}_X S)(Y, Z) := (\bar{\nabla}_X(S(Y, Z)))^{\perp} - S(\nabla_X Y, Z) - S(Y, \nabla_X Z)$.

- (9) We will denote $\nabla_X^{\perp} \xi = (\bar{\nabla}_X \xi)^{\perp}$ and $R^{\perp}(X, Y)\xi = -\nabla_X^{\perp} \nabla_Y^{\perp} \xi + \nabla_Y^{\perp} \nabla_X^{\perp} \xi + \nabla_{[X, Y]}^{\perp} \xi$. Verify the Ricci equation

$$\bar{Rm}(X, Y, \xi, \zeta) = \langle R^{\perp}(X, Y)\xi, \zeta \rangle + \langle S_{\xi} S_{\zeta} X - S_{\zeta} S_{\xi} X, Y \rangle.$$

[These three equations are the fundamental equations in submanifold geometry.]

2.10 Riemannian Geometry of Lie Groups: Part II

Exercise 2.10. Let G be a Lie group endowed with a bi-invariant Riemannian metric g . Suppose $X, Y, Z \in \mathfrak{g}$ are left-invariant vector fields on G .

- (1) Prove: $\langle [X, Y], Z \rangle = \langle [Y, Z], X \rangle$.
- (2) Prove: $\nabla_X X = 0$.
- (3) Prove: $\nabla_X Y = \frac{1}{2}[X, Y]$.
- (4) Prove: $R(X, Y)Z = -\frac{1}{4}[[X, Y], Z]$.
- (5) Let X, Y be orthonormal, and $\Pi_p \in T_p M$ be the 2-dim plane spanned by X_p, Y_p . Prove: $K(\Pi_p) = \frac{1}{4}\|[X, Y]\|^2$.
- (6) Prove: G has positive Ricci curvature if the center of G is discrete.

2.11 Riemannian Submersions

Exercise 2.11. Let $(\bar{M}, g_{\bar{M}})$ and (M, g_M) be Riemannian manifolds, and $f : \bar{M} \rightarrow M$ a submersion. For any $p \in \bar{M}$, we will call $V_p = \ker(df_p) \subset T_p \bar{M}$ the vertical space at p , and $H_p = V_p^\perp \subset T_p \bar{M}$ the horizontal space at p . For any vector $\bar{X}_p \in T_p \bar{M}$, we will denote $\bar{X}^v$ the vertical component of $\bar{X}$. Note that df_p , when restricted to H_p , is a linear isomorphism. For any $X \in \Gamma^\infty(TM)$, its horizontal lift is the horizontal vector field $\bar{X}$ defined by $df_p(\bar{X}_p) = X_{f(p)}$. The submersion f is said to be a Riemannian submersion if for any $p \in \bar{M}$, $df_p : H_p \rightarrow T_{f(p)}M$ is a linear isometry.

- (1) Prove: For any $X, Y \in \Gamma(TM)$,

$$\bar{\nabla}_{\bar{X}} \bar{Y} = \overline{\nabla_X Y} + \frac{1}{2}[\bar{X}, \bar{Y}]^v.$$

- (2) Prove: For any $X, Y, Z, W \in \Gamma(TM)$,

$$\overline{Rm}(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) = Rm(X, Y, Z, W) - \frac{1}{4}\langle [\bar{X}, \bar{Z}]^v, [\bar{Y}, \bar{W}]^v \rangle + \frac{1}{4}\langle [\bar{Y}, \bar{Z}]^v, [\bar{X}, \bar{W}]^v \rangle - \frac{1}{2}\langle [\bar{Z}, \bar{W}]^v, [\bar{X}, \bar{Y}]^v \rangle.$$

- (3) Let Π be a plane spanned by orthogonal unit vectors $X_p, Y_p \in T_p M$ and $\bar{\Pi}$ the plane spanned by $\bar{X}_p, \bar{Y}_p$. Prove: $K(\Pi) = K(\bar{\Pi}) + \frac{3}{4}|[\bar{X}_p, \bar{Y}_p]^v|^2$.

2.12 Gauss-Bonnet-Chern in Dimension 4

Exercise 2.12. The famous Gauss-Bonnet-Chern formula says that if (M, g) is an orientable closed Riemannian manifold of dimension $m = 2k$, then

$$\frac{1}{2^m \pi^{m/2} (m/2)!} \int_M \Omega = \chi(M),$$

where $\chi(M)$ is the Euler characteristic of M , and Ω is the following m -form

$$\Omega = \sum_{\sigma \in S_m} (-1)^\sigma \Omega_{\sigma(2)}^{\sigma(1)} \wedge \Omega_{\sigma(4)}^{\sigma(3)} \wedge \cdots \wedge \Omega_{\sigma(m)}^{\sigma(m-1)},$$

where S_m is the permutation group of $(1, \dots, m)$, and Ω_j^i is the curvature two form associated to any orthonormal basis.

- (1) For $m = 2$, deduce $\frac{1}{2\pi} \int_M K dx = \chi(M)$.
- (2) For $m = 4$, deduce $\frac{1}{32\pi^2} \int_M (|Rm|^2 - 4|Ric|^2 + S^2) dx = \chi(M)$.

- (3) Prove: If (M, g) is an Einstein manifold of dimension 4, then $\chi(M) \geq 0$, and the equality holds if and only if (M, g) is flat.
- (4) Prove: If (M, g) is a locally conformally flat manifold of dimension 4, then

$$\chi(M) = \frac{1}{16\pi^2} \int_M \left(-|E|^2 + \frac{1}{12} S^2 \right) dx.$$

- (5) Prove: One can find an orthonormal frame so that $R_{1213} = R_{1214} = R_{1223} = R_{1224} = R_{1323} = R_{1314} = 0$.
- (6) Prove: If M is a compact orientable Riemannian manifold of dimension 4 which admits a metric of positive sectional curvature, then $\chi(M) > 0$.

Problem Sheet 3

Geodesics

3.1 Examples of Geodesics

Exercise 3.1.

- (1) Consider (S^2, g_{round}) .
 - (a) Show that the “equator” $\gamma(\theta) = (\cos \theta, \sin \theta, 0)$ is a geodesic.
 - (b) Show that the “meridian” $\gamma(z) = (\sqrt{1 - z^2}, 0, z)$ is not a geodesic, then find a correct parametrization so that it becomes a geodesic.
- (2) Describe the relations between the geodesics on the product Riemannian manifold $(M_1 \times \cdots \times M_k, g_1 \times \cdots \times g_k)$ with the geodesics on (M_i, g_i) ’s.
- (3) Consider the hyperbolic space $\mathbb{H}^2 = \{(x, y) \mid y > 0\}$ endowed with the hyperbolic metric $g = \frac{1}{y^2}(dx \otimes dx + dy \otimes dy)$.
 - (a) Prove: The maps $\varphi(x, y) = (-x, y)$ and $\phi(x, y) = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}\right)$ are isometries.
 - (b) Deduce that the upper unit semicircle and the positive y -axis are geodesics.
 - (c) Prove: The maps $(x, y) \mapsto (x + a, y)$ and $(x, y) \mapsto (bx, by)$ are isometries for any a and any $b > 0$.
 - (d) Figure out all geodesics on $\mathbb{H}^2$. [Write down the correct parametrization.]
 - (e) Is $(\mathbb{H}^2, g)$ complete?
- (4) Describe geodesics on Lie groups (endowed with a bi-invariant metric).

3.2 Torsion Free Connection v.s. Geodesics

Exercise 3.2. Let M be a smooth manifold and let ∇, ∇' be two linear connections on M . For any vector fields $X, Y \in \Gamma(TM)$, define

$$A(X, Y) = \nabla_X Y - \nabla'_X Y.$$

- (1) Prove: A is a $(0, 2)$ -tensor.
- (2) Prove:
 - (a) ∇ and ∇' have the same torsion if and only if
$$A(X, Y) = A(Y, X), \quad \forall X, Y \in \Gamma^\infty(TM).$$
 - (b) ∇ and ∇' have the same geodesics if and only if
$$A(X, Y) = -A(Y, X), \quad \forall X, Y \in \Gamma^\infty(TM).$$
 - (c) For any linear connection ∇ , there is a unique torsion free connection ∇' with the same geodesics.
- (3) Suppose g is a Riemannian metric on M , and suppose ∇ is a g -compatible linear connection.
 - (a) Prove: ∇' is g -compatible if and only if
$$g(A(X, Y), Z) = -g(Y, A(X, Z)), \quad \forall X, Y, Z \in \Gamma^\infty(TM).$$
 - (b) Give another proof of the uniqueness part of the fundamental theorem in Riemannian geometry.

3.3 Examples of the Exponential Maps

Exercise 3.3. For each of the following, write down the exponential map $\exp_p$:

- (1) $M = \mathbb{R}^n$ with standard g_0 , and $p = 0$ the origin.
- (2) $M = \mathbb{H}^2$ (the Poincaré upper plane), and $p = (0, 1)$.
- (3) $M = S^2$ with the standard round metric, and p = the North pole.
- (4) $M = S^1 \times \mathbb{R}$ with the standard product metric, and $p = ((1, 0), 0)$.
- (5) M be the standard paraboloid defined by $z = x^2 + y^2$ in $\mathbb{R}^3$, and $p = (0, 0, 0)$.
- (6) $M = G$ (equipped with bi-invariant Riemannian metric) a matrix Lie group, and $p = I$.

3.4 Riemannian Geometry of Surfaces of Revolution

Exercise 3.4. Consider the surface of revolution

$$S(x, \theta) = (y(x) \cos \theta, y(x) \sin \theta, z(x)), \quad a < x < b, 0 < \theta < 2\pi,$$

endowed with the canonical metric

$$g = (\dot{y}(x)^2 + \dot{z}(x)^2)dx \otimes dx + y(x)^2 d\theta \otimes d\theta.$$

For simplicity, in what follows we will assume $\dot{y}(x)^2 + \dot{z}(x)^2 = 1$.

- (1) What is the geometric meaning of $\dot{y}(x)^2 + \dot{z}(x)^2 = 1$?
- (2) What is the Gaussian curvature at each point?
- (3) Write down the equations for a curve $\gamma : x = x(t), \theta = \theta(t)$ to be a geodesic.
- (4) Check that the meridians $\gamma : x(t) = t, \theta(t) = c$ are geodesics.
- (5) Find the condition for a parallel $\gamma : x(t) = c, \theta(t) = t$ to be a geodesic.
- (6) Now suppose $\gamma : x = x(t), \theta = \theta(t)$ be a normal geodesic that is neither a meridian nor a parallel. At any point $\gamma(t)$, let $\beta(t)$ be the angle between γ and the parallel that passes $\gamma(t)$. Prove: $\cos \beta(t) = y(x(t))\dot{\theta}(t)$.
- (7) Prove the Clairaut relation: $y(t) \cos \beta(t) = \text{constant}$ along γ

3.5 Riemannian Submersion: Part II

Exercise 3.5. Let $f : (\overline{M}, g_{\overline{M}}) \rightarrow (M, g_M)$ be a Riemannian submersion.

- (1) Let $\overline{\gamma}$ be any curve in $\overline{M}$, and $\gamma = f \circ \overline{\gamma}$. Prove: $\text{Length}(\overline{\gamma}) \geq \text{Length}(\gamma)$.
- (2) Let γ be a geodesic of M emanating from $\overline{p}$ which is a horizontal lift of γ .
- (3) Conversely, show that if $\overline{\gamma}$ is a geodesic in $\overline{M}$ such that $\dot{\overline{\gamma}}(0)$ is horizontal, then $\dot{\overline{\gamma}}(t)$ is horizontal for all t , and $\gamma = f \circ \overline{\gamma}$ is a geodesic in M .

3.6 Completeness of Riemannian Metric: Examples

Exercise 3.6.

- (1) Suppose (M, g) satisfies “there exists $\varepsilon_0 > 0$ so that for any $p \in M$ and $X_p \in T_p M$, there is a normal geodesic $\gamma : [0, \varepsilon_0] \rightarrow M$ with $\gamma(0) = p, \dot{\gamma}(0) = X_p$ ”. Prove: (M, g) is complete.
- (2) Suppose (M, g) is complete and $\tilde{g}$ is another Riemannian metric on M with $\tilde{g} \geq g$. Prove: $(M, \tilde{g})$ is complete.
- (3) Let (M, g) be a Riemannian manifold and there exists a proper [i.e. the pre-images of compact sets are compact] Lipschitz function $f : M \rightarrow \mathbb{R}$. Prove: (M, g) is complete.

- (4) Prove: (M, g) is complete if and only if for some fixed r , the closed geodesic ball $B_r(p)$ is compact for any $p \in M$.
- (5) A Riemannian homogeneous space is a Riemannian manifold such that its isometry group acts transitively on it, i.e. for any $p, q \in M$ there exists an isometry φ of (M, g) so that $\varphi(p) = q$. Prove: Any Riemannian homogeneous space is complete.

3.7 Completeness of Riemannian Metrics II: Existence

Exercise 3.7.

- (1) Let's prove the existence of a complete Riemannian metric on any smooth manifold M . In fact, we shall prove that given any Riemannian metric g , there exists a complete Riemannian metric g' that is conformal to g . Without loss of generality, suppose M is non-compact and g is incomplete.
 - (a) Let $r(p) = \sup\{r \mid B_r(p) \text{ is compact}\}$. Prove: $0 < r(p) < \infty$ for all $p \in M$.
 - (b) Check: $|r(p) - r(q)| < \text{dist}(p, q)$, and conclude that r is continuous.
 - (c) Prove: There exists a smooth function ω on M so that $\omega(p) > \frac{1}{r(p)}$.
 - (d) Define a new metric g' on M by $g' = \omega^2 g$. Denote by $B'_r(p)$ the closed geodesic ball of radius r around p with respect to the new metric g' . Prove: $B'_p\left(\frac{1}{3}\right) \subset B_p\left(\frac{r(p)}{2}\right)$ for all p .
 - (e) Conclude that (M, g') is complete.
- (2) Next let's prove that if every Riemannian metric on M is complete, then M is compact. Equivalently, we want to construct a incomplete Riemannian metric on any non-compact smooth manifold. Again we shall prove a stronger result: for any (M, g) with M non-compact, there is a Riemannian metric g' conformal to g which is incomplete. Without loss of generality, assume g is complete.
 - (a) Fix $p \in M$ and let ω be a smooth function on M so that $\omega(q) > \text{dist}(p, q)$. [The existence is the same as (1)(c) above.] Let $g' = e^{-2\omega}g$. Prove: Under g' the new distance $\text{dist}'(p, q) < 1$ for all q .
 - (b) Conclude that (M, g') is incomplete.

3.8 Rays in Complete Noncompact Riemannian Manifolds

Exercise 3.8. Let (M, g) be a complete noncompact Riemannian manifold.

- A normal geodesic $\gamma : [0, +\infty) \rightarrow M$ is called a ray if $\text{dist}(\gamma(a), \gamma(b)) = |a - b|$ for any $a, b \geq 0$.
 - A normal geodesic $\gamma : (-\infty, \infty) \rightarrow M$ is called a geodesic line if $\text{dist}(\gamma(a), \gamma(b)) = |a - b|$ for any $a, b \in \mathbb{R}$.
- (1) Prove: From any point $p \in M$ there exists a ray so that $\gamma(0) = p$.
 - (2) We say M is discontinuous at infinity if there exists a compact subset K in M so that $M \setminus K$ contains at least two non-compact connected components. Prove: If (M, g) is complete and M is discontinuous at infinity, then M contains a geodesic line.
 - (3) Construct a complete noncompact Riemannian manifold on which there is no geodesic line.
 - (4) Now let γ_1, γ_2 be two geodesic rays on M . We say γ_1, γ_2 are asymptotic if there exists $C \in \mathbb{R}$ so that $\text{dist}(\gamma_1(t), \gamma_2(t)) < C$ for all $t \geq 0$. This defines an equivalent relation on the set of geodesic rays. We will denote the set of equivalent classes by $M(\infty)$, and call it the sphere at infinity of M . Figure out $M(\infty)$ for $M = \mathbb{R}^n$, $S^1 \times \mathbb{R}^1$ and $\mathbb{H}^2$, each equipped with the standard Riemannian metric.

3.9 Existence of Variation

Exercise 3.9. Let $V(t)$ a piecewise smooth vector field along a smooth curve $\gamma : [a, b] \rightarrow M$.

- (1) Generalize the concept of variation to allow piecewise variation field.
- (2) Prove: There exists a variation of γ whose variation field is V .
- (3) Prove: If $V(a) = 0$, $V(b) = 0$, then the variation could be chosen to be proper.

3.10 Second Variation of Length

Exercise 3.10. Let $\gamma : [a, b]_t \rightarrow M$ be a smooth curve

$$f : [a, b] \times (-\varepsilon, \varepsilon)_v \times (-\delta, \delta)_w \rightarrow M$$

be a smooth variation of γ with two parameters v, w . Denote $\gamma_{v,w}(t) = f(t, v, w)$ and

$$f_t(t, v, w) = df_{(t,v,w)} \left(\frac{\partial}{\partial t} \right), \quad f_v(t, v, w) = df_{(t,v,w)} \left(\frac{\partial}{\partial v} \right), \quad f_w(t, v, w) = df_{(t,v,w)} \left(\frac{\partial}{\partial w} \right)$$

as usual and let

$$V(t) = df_{(t,0,0)} \left(\frac{\partial}{\partial v} \right), \quad W(t) = df_{(t,0,0)} \left(\frac{\partial}{\partial w} \right)$$

be the corresponding variation fields of the two parameter directions.

(1) Prove:

$$\begin{aligned} \frac{\partial^2}{\partial w \partial v} L(\gamma_{v,w}) &= \int_a^b \left\{ \langle \tilde{\nabla}_{\partial/\partial t} f_v, \tilde{\nabla}_{\partial/\partial t} f_w \rangle + \langle R(f_t, f_w) f_t, f_v \rangle + \langle \tilde{\nabla}_{\partial/\partial t} \tilde{\nabla}_{\partial/\partial w} f_v, f_t \rangle \right. \\ &\quad \left. - \frac{1}{|f_t|^2} \langle \tilde{\nabla}_{\partial/\partial t} f_v, f_t \rangle \langle \tilde{\nabla}_{\partial/\partial t} f_w, f_t \rangle \right\} dt \end{aligned}$$

(2) Let γ be a normal geodesic, prove

$$\left. \frac{\partial^2}{\partial w \partial v} L(\gamma_{v,w}) \right|_{(0,0)} = \int_a^b (\langle \nabla_{\dot{\gamma}} V, \nabla_{\dot{\gamma}} W \rangle + \langle R(\dot{\gamma}, W) \dot{\gamma}, V \rangle - \dot{\gamma} \langle \dot{\gamma}, V \rangle \dot{\gamma} \langle \dot{\gamma}, W \rangle) dt + \langle \tilde{\nabla}_W f_v, \dot{\gamma} \rangle|_a^b.$$

(3) Let $V^\perp, W^\perp$ be the orthogonal component of V, W with respect to $\dot{\gamma}$, i.e.

$$V^\perp = V - \langle V, \dot{\gamma} \rangle \dot{\gamma}, \quad W^\perp = W - \langle W, \dot{\gamma} \rangle \dot{\gamma}.$$

Show that

$$\left. \frac{\partial^2}{\partial w \partial v} L(\gamma_{v,w}) \right|_{(0,0)} = \int_a^b (\langle \nabla_{\dot{\gamma}} V^\perp, \nabla_{\dot{\gamma}} W^\perp \rangle + \langle R(\dot{\gamma}, W^\perp) \dot{\gamma}, V^\perp \rangle) dt + \langle \tilde{\nabla}_W f_v, \dot{\gamma} \rangle|_a^b.$$

Problem Sheet 4

Jacobi Field

4.1 Jacobi Fields for Manifolds with Constant Sectional Curvature Along γ

Exercise 4.1. Let (M, g) be a Riemannian manifold, and $\gamma : [0, l] \rightarrow M$ a normal geodesic, where we assume $l \neq k\pi/\sqrt{\kappa}$ if $\kappa > 0$. Suppose M has constant sectional curvature κ along γ , i.e. $K(\Pi_{\gamma(t)}) = \kappa$ for any 2-dimensional plane $\Pi_{\gamma(t)} \ni \dot{\gamma}(t)$.

- (1) Prove: The Jacobi field V along γ with $V(0) = 0$ and $V(l) = X_l \in (\dot{\gamma}(l))^\perp$ is

$$V(t) = \frac{sn_\kappa(t)}{sn_\kappa(l)} X(t),$$

where X is the parallel vector field along γ with $X(l) = X_l$.

- (2) Given any $X_0 \in T_{\gamma(0)}M$ and $X_l \in T_{\gamma(l)}M$, find the Jacobi field with $V(0) = X_0$, $V(l) = X_l$.

4.2 Characterizing Constant Curvature via Jacobi Field

Exercise 4.2. We say a vector field Y along γ is almost parallel if there is a smooth function f so that $Y(t) = f(t)X(t)$, where X is parallel along γ . As we have seen, any normal Jacobi field along a geodesic on a constant curvature space is almost parallel.

- (1) Suppose $\gamma : [0, l] \rightarrow M$ is a geodesic in (M, g) with $\gamma(0) = p$, so that any normal Jacobi field along γ is almost parallel. Let $V \subset T_pM$, and V_0 a small neighborhood of 0 in V such that $\exp_p : V_0 \rightarrow N = \exp_p(V_0)$ is a diffeomorphism. Suppose $\gamma([0, l]) \subset N$. Prove: $P_{0,l}^\gamma(V) = T_{\gamma(l)}N$.
- (2) Suppose for any geodesic γ with $\gamma(0) = p$, any normal Jacobi field along γ is almost parallel. Prove: for any pair wise orthogonal vectors $u, v, w \in T_pM$, $\langle R(u, v)u, w \rangle = 0$.
- (3) Suppose $m \geq 3$. Prove: If any normal Jacobi field along any geodesic in M is almost parallel, then M has constant sectional curvature.

4.3 Square of Distance in Normal Coordinates

Exercise 4.3. Consider two geodesics $\gamma_1(t) = \exp_p(tv)$ and $\gamma_2(t) = \exp_p(tw)$ emanating from p . Let $g(s) = d^2(\gamma_1(s), \gamma_2(s))$. To estimate $L(s)$ for s small, consider the variation

$$f(t, s) = \sigma_s(t) := \exp_{\gamma_1(s)}(t \exp_{\gamma_1(s)}^{-1}(\gamma_2(s))).$$

Let $f_t(t, s) = df_{t,s}(\partial/\partial t)$ and $f_s(t, s) = df_{t,s}(\partial/\partial s)$ as usual. Note that for s small, σ_s is the minimizing geodesic from $\gamma_1(s)$ to $\gamma_2(s)$, and $g(s) = \|\dot{\sigma}_s(t)\|^2 = \|f_t(t, s)\|^2$. [In what follows, although we use usual notation for the connection, they should be understood as the induced connection.]

- (1) Show that $f_t(t, 0) = 0$, $f_s(0, s) = \dot{\gamma}_1(s)$, $f_s(1, s) = \dot{\gamma}_2(s)$.
- (2) Show that $(\nabla_{\partial/\partial t} f_t)_{(t,s)} = 0$, $(\nabla_{\partial/\partial s} f_s)_{(0,s)} = 0$, $(\nabla_{\partial/\partial s} f_s)_{(1,s)} = 0$.

- (3) Show that for each fixed s , $f_s(t, s)$ is a Jacobi field along γ_s , and $(\nabla_{\partial/\partial t} \nabla_{\partial/\partial t} f_s)_{t,0} = 0$. Conclude that $f_s(t, 0)$ is linear and thus $f_s(t, 0) = v + t(w - v)$.
- (4) Show that $g'(0) = 0$, $g''(0) = 2|v - w|^2$.
- (5) Show that $(\nabla_{\partial/\partial t} \nabla_{\partial/\partial t} \nabla_{\partial/\partial s} f_s)_{(t,0)} = 0$, which implies $(\nabla_{\partial/\partial s} f_s)_{(t,0)}$ is linear in t . Conclude that $(\nabla_{\partial/\partial s} f_s)_{(t,0)} = 0$, $(\nabla_{\partial/\partial t} \nabla_{\partial/\partial s} f_s)_{(t,0)} = 0$, $(\nabla_{\partial/\partial s} \nabla_{\partial/\partial s} f_t)_{(t,0)} = 0$, and $g'''(0) = 0$.
- (6) Show that $g'''(0) = 8Rm(v, w, v, w)$.
- (7) Conclude that $g(s) = |v - w|^2 s^2 + \frac{1}{3} Rm(v, w, v, w) s^4 + O(s^5)$ and thus

$$d(\gamma_1(s), \gamma_2(s)) = |v - w|s + \frac{1}{6} \frac{Rm(v, w, v, w)}{|v - w|} s^3 + O(s^4).$$

4.4 Expansion of Metric in Normal Coordinates

Exercise 4.4.

- (1) Prove: In a Riemannian normal coordinates, near p we have

$$g_{ij} = \delta_{ij} - \frac{1}{3} R_{ikjl} x^k x^l - \frac{1}{6} R_{ikjl;r} x^k x^l x^r + O(|x|^4).$$

What can you say about the coefficients of higher order terms?

- (2) Expand $\det(g_{ij})$ up to order 3 near p .
- (3) Prove Bianchi Identity II using normal coordinates.
- (4) Prove: A chart is a Riemannian normal coordinate system if and only if for any i , $g_{ij}x^j = x^i$.

4.5 Locally Symmetric Space

Exercise 4.5. Let (M, g) be a locally symmetric space, i.e. if $\nabla_X R = 0$ for all $X \in \Gamma^\infty(TM)$. Let $\gamma : [0, a] \rightarrow M$ be a geodesic in M with $p = \gamma(0)$, $X_p = \dot{\gamma}(0)$.

- (1) Let X, Y, Z be vector fields that are parallel along γ . Prove: $R(X, Y)Z$ is also parallel along γ .
- (2) Define a linear transformation $K_{X_p} : T_p M \rightarrow T_p M$ by

$$K_{X_p}(Y_p) = R(X_p, Y_p)X_p.$$

Prove: K_{X_p} is self-adjoint.

- (3) Let $\lambda_1, \dots, \lambda_m$ be eigenvalues of K_{X_p} , with corresponding eigenvectors $e_1, \dots, e_m \in T_p M$. Let $e_j(t)$ be the parallel transport of e_j along γ . Prove: For all $t \in [0, \infty)$, we have

$$K_{\dot{\gamma}(t)}(e_i(t)) = \lambda_i e_i(t), \quad i = 1, \dots, m.$$

- (4) Let $X(t) = X^i(t)e_i(t)$ be a Jacobi field along γ . Show that the Jacobi equation becomes

$$\ddot{X}^i(t) + \lambda_i X^i = 0, \quad i = 1, \dots, m.$$

- (5) Conclude that the conjugate points of p along γ are given by

$$\gamma\left(\frac{\pi k}{\sqrt{\lambda_i}}\right), \quad k = 1, 2, \dots,$$

where λ_i 's are positive eigenvalues of K_{X_p} .

4.6 More on Cut Locus

Exercise 4.6. Let (M, g) be complete.

- (1) Prove: The function $f : SM \rightarrow \mathbb{R} \cup \{\infty\}$ defined by

$$f(X_p) = \begin{cases} t_0, & \text{if } \gamma_{p, X_p}(t_0) \text{ is the cut point of } p \text{ along } \gamma, \\ +\infty, & \text{if } p \text{ has no cut point along } \gamma_{p, X_p}. \end{cases}$$

is continuous.

- (2) For any $p \in M$, $\text{Cut}(p)$ is closed.
- (3) M is compact if and only if $\text{Cut}(p)$ is nonempty and compact for any $p \in M$.
- (4) Let $\Sigma(p) = \{tX_p \mid X_p \in S_p M, 0 \leq t < f(X_p)\}$. Prove: $\exp_p : \Sigma(p) \rightarrow \exp_p(\Sigma(p))$ is a diffeomorphism.
- (5) Prove: $M = \exp_p(\Sigma(p)) \cup \text{Cut}(p)$ and $\exp_p(\Sigma(p)) \cap \text{Cut}(p) = \emptyset$.
- (6) We call a point $q \in M$ a regular cut point if there exists at least two minimal geodesic from p to q . Prove: The set of regular cut points is a dense subset of $\text{Cut}(p)$.

4.7 Smoothness of Distance Function

Exercise 4.7. For any $p \in M$, consider the distance square function

$$f(q) = \frac{1}{2} \text{dist}(p, q)^2.$$

- (1) For $(M, g) = (S^m, g_{S^m})$ the standard sphere, is f a smooth function?
- (2) Argue that f is smooth on $M \setminus \text{Cut}(p)$.
- (3) For any $q \in M \setminus \text{Cut}(p)$, find $(\nabla f)(q)$.
- (4) For any $q \in M \setminus \text{Cut}(p)$ and $Y_q \in T_q M$, let X be a Jacobi field along γ^q , the minimal geodesic from p to q , so that $X(0) = 0$, $X(\text{dist}(p, q)) = Y_q$. Prove: $\nabla^2 f(Y_q, Y_q) = \text{dist} \langle \nabla_{\dot{\gamma}^q(\text{dist}(p, q))} X, X(\text{dist}(p, q)) \rangle$.
- (5) Prove: f is not C^1 function at regular cut points.
- (6) Can f be everywhere smooth on M if M is compact?

4.8 Convex Functions on Riemannian Manifolds

Exercise 4.8. • Let (M, g) be a Riemannian manifold. A function $f : M \rightarrow \mathbb{R}$ is said to be a convex function if for any geodesic $\gamma : [a, b] \rightarrow M$, the function $f \circ \gamma : [a, b] \rightarrow \mathbb{R}$ is convex.

- (1) Prove: If f is a convex function on M , then for any $c \in \mathbb{R}$, the sublevel set $M_c = \{p \in M \mid f(p) < c\}$ is a totally convex subset of M .
- (2) Prove: If f is smooth, then f is convex if and only if its Hessian $\nabla^2 f$ is positive semidefinite.
- (3) Let $p \in M$ be an arbitrary point, and $d_p(q) = \text{dist}(p, q)$ is the distance function from p . Prove: There exists an neighborhood U of p so that the distance square function d_p^2 is convex on (U, g) .
- Now suppose (M, g) is a complete simply-connected Riemannian manifold with non-positive sectional curvature.
- (4) Prove: the distance square function

$$d^2 : M \times M \rightarrow \mathbb{R},$$

$$d^2(p, q) = [\text{dist}(p, q)]^2$$

is convex on $M \times M$.

- (5) Conclude that for any $p \in M$, the function d_p^2 is a convex function on M .

Topics of Final Projects

1. Myers-Steenrod theorem: The isometry group of any Riemannian manifold is a Lie group.
2. Cartan-Janet local isometry theorem: Any analytic m -dimensional Riemannian manifold admits an analytic local isometric embedding into $\mathbb{R}^{m(m+1)/2}$.
3. Gromov compactness theorem: Let (M_i, g_i) be a sequence of closed Riemannian manifolds with $\text{Ric} \geq (m-1)c$ and $\text{diam}(M, g) \geq \delta$. Then there is a subsequence (M_{k_i}, g_{k_i}) that converges, with respect to the Gromov-Hausdorff distance, to a compact metric space.
4. Lusternik-Fet closed geodesic theorem: Any compact Riemannian manifold (M, g) admits a nontrivial closed geodesic.
5. Weyl-Schouten theorem: Let (M, g) be a Riemannian manifold. If $m \geq 4$, then (M, g) is locally conformally flat if and only if the Weyl tensor $W = 0$. If $m = 3$, then (M, g) is locally conformally flat if and only if $(\nabla_X A)(Y, Z) - (\nabla_Y A)(X, Z) = 0, \forall X, Y, Z \in \Gamma^\infty(TM)$, where A is the Schouten tensor.
6. Cheeger-Gromoll soul theorem: For any complete noncompact connected Riemannian manifold (M, g) with non-negative sectional curvature, there exists a totally convex compact submanifold without boundary (known as the soul) whose normal bundle is diffeomorphic to M .
7. Cartan-Ambrose-Hicks theorem: Let (M, g) and $(\widetilde{M}, \widetilde{g})$ be complete Riemannian manifolds and suppose M is simply connected. Suppose $L : (T_p M, g_p) \rightarrow (T_{\widetilde{p}} \widetilde{M}, \widetilde{g}_{\widetilde{p}})$ is a linear isometry, and assume that for any broken geodesic (i.e. a curve that is piecewise minimizing geodesic) γ , we have, as in Cartan's local isometry theorem, that $L_q(R(u, v)w) = \widetilde{R}(L_q(u), L_q(v))L_q(w)$ for any $u, v, w \in T_q M$. Then if two broken geodesics γ_1, γ_2 beginning at p in M have the same endpoint, the corresponding broken geodesics in $\widetilde{M}$ also have the same end point. Moreover, the resulting natural map $\Phi : M \rightarrow \widetilde{M}$ is a locally isometry (and hence covering map).
8. The Gauss-Bonnet-Chern theorem: If (M, g) is an orientable closed Riemannian manifold of dimension $m = 2k$, then $\frac{1}{2^m \pi^{m/2} (m/2)!} \int_M \Omega = \chi(M)$.
9. Korn-Lichtenstein theorem on the existence of isothermal coordinates: On any 2-dimensional Riemannian manifold (M, g) , there exists coordinates x, y and smooth function $u = u(x, y)$ so that $g = e^{2u}(dx \otimes dx + dy \otimes dy)$.
10. Ambrose-Singer holonomy theorem: As Lie subalgebra of the Lie algebra of endomorphisms of $T_p M$, the Lie algebra of the holonomy group with base point p is the subalgebra generated by the endomorphisms of the form $(P^\gamma)^{-1} \circ R(P^\gamma u, P^\gamma v) \circ P^\gamma$, where R is the curvature endomorphism, and γ is a curve starting at p .
11. Curvature of left-invariant metrics on compact Lie group.
12. Riemannian geometry of $\mathbb{CP}^m$.
13. Riemannian geometry of the Grassmann manifold $G(k, n)$.
14. Riemannian geometry of TM (with Sasaki metric).
15. Riemannian geometry of isoparametric hypersurface in space form.