Lorentzian Geometry Problem Sheets

Dr. Onirban Islam & Dr. Rubens Longhi

Contents

Problem Sheet 1

Foundations & Fundamental Computations

Exercise 1.1. Let $d \geqslant 1$ be an integer and $\mathbb{R}^{d+1}$ endowed with the scalar product

$$\begin{aligned} \cdot : \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} &\to \mathbb{R}, \\ (x, y) &\mapsto x \cdot y := -x^0 y^0 + \sum_{i=1}^{d} x^i y^i, \end{aligned} \tag{1.1}$$

where $x = (x^0, x^i) \in \mathbb{R} \times \mathbb{R}^d$. Show that the pair $(\mathbb{R}^{d+1}, \cdot)$ is a Lorentzian manifold and the standard basis $\{e_0, \ldots, e_n\}$ for $\mathbb{R}^{d+1}$ is Lorentz-orthonormal.

Exercise 1.2. Consider the matrix $A := \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ with elements $a, b, c \in \mathcal{C}^\infty(\mathbb{R}^2)$. Let $(\mathbb{R}^2, g)$ be the Lorentzian manifold where $g := A_{ij} \mathrm{d}x^i \mathrm{d}x^j$ in Cartesian coordinates $x = (x^0, x^1)$.

(1) Under which coordinates on a, b, c, the metric g is Riemannian and non-degenerate?

(2) Compute the components of g in polar coordinates (r, θ).

(3) Calculate the inverse of g in both Cartesian coordinates and polar coordinates.

Suppose that ∇ is the Levi-Civita connection compatible with g. Compute the following quantities in Cartesian coordinates in terms of a, b, c:

(4) Christoffel symbols of ∇.

(5) Components of the Riemann curvature tensor.

(6) Sectional curvature of the manifold.

Exercise 1.3. Assume that m is a positive parameter.

(1) Let $N := \mathbb{R} \times ((0, 2m) \cup (2m, \infty)) \times \mathbb{S}^2$ be a manifold endowed with the metric

$$h := -\left(1 - \frac{2m}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r^2 + r^2 h_{\mathbb{S}^2}, \tag{1.2}$$

where $N \ni x = (t, r, \cdot)$, $r \neq 2m$, and $h_{\mathbb{S}^2}$ is the metric of the round unit 2-sphere $\mathbb{S}^2$. Is the pair (N, h) a Lorentzian manifold? [Hint: scrutinise if h is a well-defined smooth Lorentzian metric on N.]

(2) Let $M := \{(x^1, x^2) \in \mathbb{R} \times \mathbb{R} \mid x^1 x^2 < 1\} \times \mathbb{S}^2$ be a manifold endowed with the metric

$$g := -32m^3 \frac{\mathrm{e}^{-\frac{r}{2m}}}{r} \mathrm{d}x^1 \mathrm{d}x^2 + r^2 h_{\mathbb{S}^2}, \tag{1.3}$$

where r is a non-negative function of (x^1, x^2) implicitly defined by

$$x^1 x^2 = \left(1 - \frac{r}{2m}\right) \mathrm{e}^{\frac{r}{2m}}$$

and $h_{\mathbb{R}^2}$ is as before. Is the pair (M, g) a Lorentzian manifold. [Hint: You can use the same strategy as before.]

Exercise 1.4. Consider the matrix $A = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$, where a, b, $c \in \mathcal{C}^\infty(\mathbb{R}^2)$ and the pseudo-Riemannian manifold $(\mathbb{R}^2, g)$, where $g = A_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j$ in Cartesian coordinates $x = (x^1, x^2) \in \mathbb{R}^2$.

(1) Under which conditions on a, b, c is g a Riemannian and non-degenerate metric?

(2) Compute the components of g in polar coordinates (r, θ).

(3) Calculate the inverse matrix A^{-1}.

Let ∇ be the Levi-Civita connection compatible with g.

(4) Calculate the Christoffel symbols Γ^k_{ij} of ∇ in Cartesian coordinates in terms of a, b, c.

(5) Compute the components of the Riemann curvature tensor R^l_{ijk} in Cartesian coordinates in terms of a, b, c.

(6) Compute the sectional curvature K of the manifold in terms of a, b, c.

Exercise 1.5. Let $\phi : \mathbb{R}^2 \to \mathbb{R}^3$ be the immersion map $\phi(x, y) = (x, y, x^2 + y^2)$. Consider the Riemannian manifold $(\mathbb{R}^2, g = \phi^*\delta)$, where δ is the Euclidean metric of $\mathbb{R}^3$.

(1) Compute the components $\begin{pmatrix} a & c \\ c & b \end{pmatrix}$ of g in Cartesian coordinates.

(2) Compute the Christoffel symbols for the connection compatible with g as well as the components of the Riemann curvature tensor and the sectional curvature $K(\Sigma)$.

Exercise 1.6. Consider $\mathbb{R}^2$ endowed with the standard Euclidean metric δ.

(1) Express δ in polar coordinates (r, ϑ).

(2) Calculate the Christoffel symbols Γ^k_{ij} for the connection compatible with δ in polar coordinates.

(3) Express the geodesic equations for the polar coordinates.

(4) Check that the straight lines (with constant velocity) are solutions of the geodesic equations for the polar coordinates.

Exercise 1.7. Let $\mathbb{S}^2$ be the 2-sphere and let $\iota : \mathbb{S}^2 \to \mathbb{R}^3$ be the immersion map as the unit sphere in $\mathbb{R}^3$. We denote with δ the Euclidean metric on $\mathbb{R}^3$ and we endow $\mathbb{S}^2$ with the metric $g = \iota^*\delta$.

(1) Express g in angular coordinates (ϑ, φ).

(2) Write δ in spherical coordinates (r, ϑ, φ).

(3) Calculate the Christoffel symbols Γ^k_{ij} for δ in spherical coordinates.

(4) Express the geodesic equation for the spherical coordinates.

(5) Check that the straight lines (with constant velocity) are solutions of the geodesic equations for the spherical coordinates.

Problem Sheet 2

The Lorentz Group, $SL(2, \mathbb{C})$ and Double Cover

Let $M^{n+1} = (\mathbb{R}^{n+1}, \langle\langle \cdot, \cdot \rangle\rangle)$ be the Minkowski space and $\{e_i\}_{i=0,\dots,n}$ be the standard basis.

Exercise 2.1. Let $\Lambda \in \mathscr{L}_+^\uparrow(n+1)$ with $\Lambda e_0 = e_0$. Show that Λ identifies a spatial rotation, i.e. is of the form $\Lambda = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \mathbf{A} \end{array}\right)$ with $\mathbf{A} \in SO(n)$. Let $\Lambda \in \mathscr{L}_+^\uparrow(n+1)$ with $\Lambda e_i = e_i$ for $i = 2, \dots, n$.

Show that Λ is a Lorentz boost. This shows that $\mathscr{L}_+^\uparrow(n+1)$ consists essentially of spatial rotations and Lorentz boosts. [Probably-useless-hint: Show that $\forall x \in M^{n+1}$ we can write $x = \sum_{i=0}^{n} \varepsilon_i \langle\langle x, e_i \rangle\rangle e_i$, where $\varepsilon_i := \langle\langle e_i, e_i \rangle\rangle$.]

The next two exercises are aimed at defining the double covering $SL(2, \mathbb{C}) \to \mathscr{L}_+^\uparrow(4)$. This shows that $SL(2, \mathbb{C})$ identifies the spin group $\mathrm{Spin}(1, 3)$.

In solving the last exercise, you can take for granted the results of the previous.

Exercise 2.2. Prove that the map $M^4 \to H(2, \mathbb{C})$

$$x \mapsto x_\star := \sum_{i=0}^{3} x^i \sigma_i$$

is an isomorphism of $\mathbb{R}$-vector spaces, where $H(2, \mathbb{C})$ denotes the set of Hermitian 2×2 matrices and

$$\sigma_0 = \mathbf{1}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are so-called Pauli matrices. Prove also that $\det x_\star = -\langle\langle x, x \rangle\rangle$, $x \in M^4$.

Endow $H(2, \mathbb{C})$ with the inner product $\langle a, b \rangle := \frac{1}{2}\mathrm{tr}(ab)$, show that $\langle \sigma_1, \sigma_j \rangle = \delta_{ij}$ [Hint to ease the calculations: the Pauli matrices σ_i, for $i = 1, 2, 3$, satisfy the Clifford relations], that $x^j = \langle x_\star, \sigma_j \rangle$ for $x \in M^4$, and write down an inverse for $x \mapsto x_\star$.

Consider $K : SL(2, \mathbb{C}) \times H(2, \mathbb{C}) \to H(2, \mathbb{C})$ given by $K(A, b) = \mathrm{A}bA^*$ and show it is a well-defined group action, where $SL(2, \mathbb{C})$ is the set of 2×2 complex matrices with unit determinant.

Exercise 2.3. Let $\rho : SL(2, \mathbb{C}) \to GL(M^4)$

$$\rho(A)x := \star^{-1}(K(A, x_\star)) \in M^4$$

, where $\star^{-1}$ is the inverse of $x \mapsto x_\star$. Show that ρ is a well-defined homomorphism and

(1) Show that the matrix elements in the standard basis satisfy $\rho(A)^i{}_j = \langle \sigma_i, A\sigma_j A^* \rangle$.

(2) Show that $\rho(SL(2, \mathbb{C})) \subset \mathscr{L}_+^\uparrow(4)$. [Hint: ρ is continuous and $SL(2, \mathbb{C})$ is connected.]

(3) Show that $\ker \rho = \{\mathbf{1}, -\mathbf{1}\}$.

Now it suffices to show that ρ is surjective onto $\mathscr{L}_+^\uparrow(4)$ to imply $SL(2, \mathbb{C})$ is the double covering of $\mathscr{L}_+^\uparrow(4)$. To this, one needs to check that ρ maps to every generator of $\mathscr{L}_+^\uparrow(4)$. As an example, only check:

(4) that $\rho(A)$ corresponds to a spatial rotation, for $A = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\frac{\theta}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\frac{\theta}{2}} \end{pmatrix}$, $\theta \in \mathbb{R}$.

(5) that $\rho(B)$ is a Lorentz boost, for $B = \begin{pmatrix} \cosh\frac{\alpha}{2} & -\sinh\frac{\alpha}{2} \\ -\sinh\frac{\alpha}{2} & \cosh\frac{\alpha}{2} \end{pmatrix}$, $\alpha \in \mathbb{R}$.

Exercise 2.4. Consider the Minkowski spacetime $M^4 = (\mathbb{R}^4, \eta)$ and the group $H(2,\mathbb{C}) = \{A \in \mathrm{Mat}(2,\mathbb{C}) \mid A = A^\dagger := (A^{\mathrm{T}})^*\}$.

(1) Show that $\dim_{\mathbb{R}} H(2,\mathbb{C}) = 4$.

(2) Show that $\{\sigma_k\}_{k=0,\dots,3}$ is a basis for $H(2,\mathbb{C})$, where $\sigma_0 = \mathbf{1}$ and $\{\sigma_k\}_{k=1,\dots,3}$ are the Pauli matrices.

Define the map $M^4 \to H(2,\mathbb{C})$.

$$x \mapsto x_\star = x^i\sigma_i$$

(3) Show explicitly the action of $x \mapsto x_\star$ and prove that it is a linear isomorphism.

(4) Prove that $x^k = \frac{1}{2}\mathrm{tr}(x_\star\sigma_k)$ and $\det x_\star = -\langle\langle x, x\rangle\rangle$, for $x \in M^4$.

Define the map $M^4 \to H(2,\mathbb{C})$.

$$x \mapsto x^\star = \sum_{j=1}^{3} x^j\sigma_j - x^0\sigma_0$$

(5) Show explicitly the action of $x \mapsto x^\star$ and prove that it is a linear isomorphism.

(6) Prove that $\det x^\star = -\langle\langle x, x\rangle\rangle$, for $x \in M^4$.

(7) Prove that $x_\star y^\star = x^\star y_\star = \langle\langle x, y\rangle\rangle\mathbf{1}$, for x, $y \in M^4$.

Define $K : SL(2,\mathbb{C}) \times M^4 \to H(2,\mathbb{C})$, $K(A,x) = Ax_\star A^\dagger$, where $SL(2,\mathbb{C})$ is the set of 2×2 complex matrices with unit determinant.

(8) Show K is well-defined.

Let $L(A,x) = \star^{-1}(K(A,x)) \in \mathbb{M}^4$, where $\star^{-1}$ is the inverse of $x \mapsto x_\star$.

(9) Show that for any $A \in SL(2,\mathbb{C})$ the map $L(A,\cdot) : M^4 \to M^4$ lies in $\mathscr{L}_+^\uparrow(M^4)$.

(10) Let $\ker L = \{A \in SL(2,\mathbb{C}) \mid L(A,x) = x, \forall x \in M^4\}$. Show that $\ker L = \{\mathbf{1}, -\mathbf{1}\}$.

In other words, $SL(2,\mathbb{C})$ is the (double) universal covering of $\mathscr{L}_+^\uparrow(M^4)$.

Exercise 2.5. Let $\theta \in \mathbb{R}$ and

$$A(\alpha) = \begin{pmatrix} \cosh\frac{\alpha}{2} & -\sinh\frac{\alpha}{2} \\ -\sinh\frac{\alpha}{2} & \cosh\frac{\alpha}{2} \end{pmatrix}$$

Show that $A(\alpha) \in SL(2,\mathbb{C})$ and that $L(A(\alpha),\cdot)$ is represented by the matrix

$$\Lambda = \begin{pmatrix} \cosh\alpha & -\sinh\alpha & 0 \\ -\sinh\alpha & \cosh\alpha & 0 \\ 0 & 0 & \mathbf{1}_2 \end{pmatrix},$$

which is a Lorentz boost in the e_1 direction.

Exercise 2.6. An observer in Minkowski spacetime is a Lorentz orthonormal basis of $\mathbb{R}^{n+1}$. We say that two observer $\{e_i\}_{i=0,\dots,n}$, $\{e'_j\}_{j=0,\dots,n}$ have relative velocity $\beta \in (-1,+1)$ along the e_1 axis if

$$e'_j = \Lambda^k{}_j(\beta)e_i, \quad \Lambda(\beta) = \begin{pmatrix} \cosh\alpha & -\sinh\alpha & 0 \\ -\sinh\alpha & \cosh\alpha & 0 \\ 0 & 0 & \mathbf{1}_{n-1} \end{pmatrix}, \quad \beta = \tanh\alpha.$$

(i.e. they are related by a Lorentz boost. In this case, α is called the rapidity.)

(1) Express $\Lambda(\beta)$ in terms of β (Use the shorthand notation $\gamma = (1-\beta^2)^{-1/2}$).

Let $\{e_i\}$, $\{e'_j\}$ and $\{e''_k\}$ be three observers such that β' is the relative velocity between the $\{e_i\}$ and the $\{e'_j\}$, while β'' is the relative velocity between the $\{e'_j\}$ and $\{e''_k\}$.

(2) Find the relative velocity β between $\{e_i\}$ and $\{e''_k\}$.

Exercise 2.7. Show how the components of an anti-symmetric totally covariant 2-tensor F on $\mathbb{R}^4$ change under a Lorentz boost.

Problem Sheet 3

de Sitter and Anti-de Sitter Spacetimes

Exercise 3.1. Show that the points $(0,-1)$ and $(0,1)$ in 2-dimensional Minkowski spacetime $(\mathbb{R}^2 \setminus \{(0,0)\}, \langle\langle \cdot \mid \cdot \rangle\rangle)$ cannot be connected by any timelike geodesic. Is it possible to join these points by a lightlike or a spacelike geodesic?

Exercise 3.2. Let $M := \mathbb{S} \times \mathbb{S} \equiv \{(t,s) \mid |t| \leqslant 4\pi, 0 \leqslant x \leqslant 1\}$ be a manifold endowed with the metric $g := -\cos(t)\mathrm{d}t^2 + 2\sin(t)\mathrm{d}t\mathrm{d}x + \cos(t)\mathrm{d}x^2$.

(1) Is the pair (M,g) a Lorentzian manifold?

(2) Is (M,g) geodesically connected?

Exercise 3.3. Let $(\mathbb{R}^{n+1}, \langle\langle \cdot \mid \cdot \rangle\rangle)$ and $(\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle)$ be the $n{+}1$-dimensional Minkowski spacetime and Euclidean space, respectively.

(1) Define the n-dimensional de Sitter spacetime $(M := S_1^n(r), g) \subset (\mathbb{R}^{n+1}.\langle\langle \cdot \mid \cdot \rangle\rangle)$ where r is a fixed positive number.

(2) What is the topology of M?

(3) Argue that the geodesic completeness of $\mathbb{S}^n \subset (\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle)$ implies the geodesic connectedness.

(4) Let $p, q \in M$ be distinct non-antipodal points in the de Sitter spacetime $(S_1^n(r), g)$ such that $\langle\langle p, q \rangle\rangle \leqslant -r^2$. Is it possible to join p and q by some geodesic in M?

Exercise 3.4. Consider n-dimensional de Sitter space time $S_1^n(r) \subset \mathbb{R}^{n+1}$. Let $(t, \{y^i\}) \in \mathbb{R}^n$ be coordinates such that

$$x^0 = r\sinh\left(\frac{t}{r}\right) + \frac{|y|^2 \mathrm{e}^{\frac{t}{r}}}{2r}$$
$$x^1 = r\cosh\left(\frac{t}{r}\right) - \frac{|y|^2 \mathrm{e}^{\frac{t}{r}}}{2r}$$
$$x^i = \mathrm{e}^{\frac{t}{r}} y^i, \ \forall i = 2, \ldots, n,$$

where $|y|^2 = \sum\limits_i (y^i)^2$. These are the so-called flat coordinates.

(1) Prove that the image of this chart lies in $S_1^n(r)$. Explain why the flat coordinates do not cover the entire $S_1^n(r)$, but only the part where $x^0 + x^1 > 0$.

(2) Prove that the metric induced on $S_1^n(r)$, expressed in flat coordinate is

$$g = -\mathrm{d}t^2 + \mathrm{e}^{\frac{2t}{r}}\mathrm{d}y^2, \quad \text{where } \mathrm{d}y^2 = \sum_i (\mathrm{d}y^i)^2.$$

(3) Find the geodesic equations for a parametrization $\gamma(\lambda) = (t(\lambda), \{y^i(\lambda)\})$. To ease the calculations, recall that a curve $\gamma : (a,b) \subseteq \mathbb{R} \to S_1^n(r)$ is a geodesic if it is a stationary point of the energy functional

$$E[\gamma] = \frac{1}{2}\int_a^b g(\dot{\gamma}(\lambda), \dot{\gamma}(\lambda))\mathrm{d}\lambda.$$

(Note that $\dot{\gamma} = \dfrac{\mathrm{d}}{\mathrm{d}\lambda}\gamma$).

(4) Prove that any timelike geodesic $\gamma(\lambda)$ is past-incomplete, i.e. that if $t \to -\infty$, then the affine parameter λ goes to a finite value.

Exercise 3.5. Consider n-dimensional anti de Sitter space time $H_1^n(r) \subset \mathbb{R}^{n+1}$. The defining equation is $-(x^0)^2 - (x^1)^2 + \sum_{i=2}^{n}(x^i)^2 = -r^2$, $r > 0$. Rewrite it in the more suggestive form

$$(x^0)^2 + (x^1)^2 = r^2 + \sum_{i=2}^{n}(x^i)^2.$$

Introduce global static coordinates $(\{z^i\}, \tau, \rho) \in S^{n-2} \times \mathbb{R} \times \mathbb{R}^+$, i.e. such that

$$\begin{aligned} x^0 &= r\cosh\rho\sin\tau \\ x^1 &= r\cosh\rho\cos\tau \\ x^i &= r\sinh\rho\, z^i, \quad z^i \in S^{n-2}, \quad \sum_{i=2}^{n}(z^i)^2 = 1, \end{aligned}$$

where $\{z^i\}$ are spherical coordinates on S^{n-2}.

(1) Calculate the metric in such coordinate system. The answer is $g = r^2(-\cosh^2\rho \mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \sinh^2\rho \mathrm{d}\Omega_{n-2}^2)$, where $\mathrm{d}\Omega_{n-2}^2$ is the standard metric on S^{n-2}.

Now set $n = 4$ and let $\theta = [0, \pi/2)$ such that $\cosh\rho = \dfrac{1}{\cos\theta}$.

(2) Show that the new metric is conformal to the Einstein universe metric $h = -\mathrm{d}\tau^2 + \mathrm{d}\Omega_3^2$. (The metric g and h are conformally related if there exists a function $F > 0$ such that $g = Fh$.)

The extension of the spacetime where $\theta = \dfrac{\pi}{2}$ has the so-called conformal boundary.

Problem Sheet 4

Hypersurfaces and Classical Spacetimes

Exercise 4.1. Show that de Sitter spacetime $S_1^n(r)$ is an example of Robertson-Walker spacetime. What are S, g_S, f and the sectional curvature κ in this case?

Exercise 4.2. Consider the n-dimensional anti de Sitter spacetime $H_1^n(r) \subset \mathbb{R}^{n+1}$.

(1) Write down a diffeomorphism between $H_1^n(r)$ and $N := \mathbb{S}^1 \times B^{\mathbb{S}^{n-1}}\left(p, \frac{\pi}{2}\right)$, where $B^{\mathbb{S}^{n-1}}\left(p, \frac{\pi}{2}\right)$ is an open ball on $\mathbb{S}^{n-1}$ of radius $\frac{\pi}{2}$ centered at any point $p \in \mathbb{S}^{n-1}$.

(2) Prove that the metric g of $H_1^n(r)$ is conformal to the one on N given by $h = -h_{\mathbb{S}^1} + h_{\mathbb{S}^{n-1}}$, where $h_{\mathbb{S}^m}$ is the standard metric of $\mathbb{S}^m$.

A metric g is conformal to another metric h if there exists a function $\Omega > 0$ such that $g = \Omega^2 h$.

[Hint: consider that the function $s \mapsto \frac{1}{\cos(s)}$ is a diffeomorphism $\left[0, \frac{\pi}{2}\right) \to [1, +\infty)$.]

The manifold $\mathbb{S}^1 \times \partial B^{\mathbb{S}^{n-1}}\left(\frac{\pi}{2}\right)$ is the so-called conformal boundary of AdS spacetime.

Exercise 4.3. Recall the Schwarzschild spacetime defined in Exercise 1.3 (1), i.e. the Lorentzian manifold $M := \mathbb{R} \times ((0, 2m) \cup (2m, \infty)) \times \mathbb{S}^2$ endowed with the metric

$$g := -\left(1 - \frac{2m}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r^2 + r^2 h_{\mathbb{S}^2},$$

where $h_{\mathbb{S}^2}$ is the standard metric of $\mathbb{S}^2$ and m is a non-negative parameter.

(1) At each point $p \in M$ let $Z \subset T_pM$ be the plane spanned by $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial r}$. Calculate the sectional curvature of a plane $E \subset T_pM$ in the following cases:

(a) if E is tangential to $\mathbb{S}^2$ or if $E = Z$.

(b) if E is spanned by a vector tangential to $\mathbb{S}^2$ and by one in Z.

(2) Fix $r_0 > 2m > 0$ and consider the hypersurface $\mathscr{H}_{r_0} = \{r = r_0\}$ in M. Show that $\mathscr{H}_{r_0}$ is totally umbilic if and only if $r_0 = 3m$.

Here totally umbilic means that the Weingaten map of the hypersurface is a multiple of the identity, i.e. $\nabla_X v = \lambda X$ for all X tangent to the hypersurface where λ is a constant and v is a unit normal field along the hypersurface.

Exercise 4.4. Prove that for any Lorenzian manifold M which is foliated by spacelike hyper surfaces with normal field ν parallel to itself ($\nabla_\nu \nu = 0$)——Riemann foliation——then the shape operator W of the hypersurfaces satisfies the Riccact equation

$$R(\cdot, \nu)\nu + \nabla_\nu W + W^2 = 0.$$

Exercise 4.5. Consider the Robertson-Walker spacetime $M = \mathbb{R} \times S^3$ ($\kappa = 1$) with metric of the form $g = -\mathrm{d}t^2 + f^2(t)\mathrm{d}\Omega_3^2$, where $\mathrm{d}\Omega_3^2$ is the standard metric on S^3 (denote (ψ, θ, ϕ) the spherical coordinates on S^3). Show that there exists a coordinate transformation such that the metric can assume the form

$$g = -\mathrm{d}t^2 + f^2(t)\left[\frac{\mathrm{d}r^2}{1 - r^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)\right],$$

where (r, θ, ϕ) are new coordinates on S^3.

Exercise 4.6. In general, a 4-dimensional Robertson-Walker spacetime $M = I \times S$, with $\kappa = 0,\ 1,\ -1$ being the sectional curvature of S is endowed with the following metric in the time+spherical coordinates (t, ψ, θ, ϕ):

$$g = -\mathrm{d}t^2 + f^2(t)[\mathrm{d}\psi^2 + h^2(\kappa)\mathrm{d}\Omega_2^2], \tag{4.1}$$

where as always $\mathrm{d}\Omega_2^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2$, while $h(0) = \psi$, $h(1) = \sin\psi$, $h(-1) = \sinh\psi$.

Consider the spatial part of the metric, namely $f^2(t)[\mathrm{d}\psi^2 + h^2(\kappa)\mathrm{d}\Omega_2^2]$.

(1) Calculate the radius $\rho(t)$ of the surface Σ such that $\psi = \psi_0$ constant.

(2) Calculate the area $A(\rho)$ of Σ for the values $\kappa = 0,\ 1,\ -1$ and explain the three different behaviours.

Exercise 4.7. Prove that, with reference to the metric (4.1), for any affine parameter λ along any radial geodesic it holds $f^2 \dfrac{\mathrm{d}\psi}{\mathrm{d}\lambda} = \text{constant}$.

Problem Sheet 5

Geodesics & Conserved Quantities

Exercise 5.1. Prove that for any geodesic $\lambda \to \gamma(\lambda)$ and for any Killing vector field X ($\mathcal{L}_X g = 0$), the quantity $g(X, \dot{\gamma})$ is constant (conserved quantity) along γ.

Exercise 5.2. Consider n-dimensional anti de Sitter spacetime $H_1^n(r) \subset \mathbb{R}^{n+1}$ with coordinates $(\tau, \rho, \{\varphi\}_{S^{n-2}}) \in \mathbb{R} \times \mathbb{R}^+ \times \cdots$ endowed with the following metric we calculated in Problem Sheet 3:

$$g = r^2(-\cosh^2 \rho \mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \sinh^2 \rho \mathrm{d}\Omega_{n-2}^2),$$

where $\mathrm{d}\Omega_{n-2}^2$ is the standard metric on S^{n-2}. Find the conserved quantities along geodesics and prove that for a lightlike radial geodesic it takes an infinite amount of proper time (affine parameter λ) to reach $\rho \to \infty$.

Exercise 5.3. Now consider AdS for $n = 4$ and let $\theta \in [0, \pi/2)$ such that $\cosh \rho = \dfrac{1}{\cos \theta}$. Recall that in these coordinates the metric is

$$g = \frac{r^2}{\cos \theta}(-\mathrm{d}\tau^2 + \mathrm{d}\theta^2 + \mathrm{d}\Omega_2^2).$$

The spacetime has the so-called conformal boundary. Find the conserved quantities along geodesics and show that for a lightlike radial geodesic it takes an infinite amount of proper time (affine parameter λ) to reach the conformal boundary $\theta \to \pi/2$. [Hint: You may use some first order approximation when $\theta \to \pi/2$ to get an explicit result.]

Exercise 5.4. Consider the Schwarzschild spacetime endowed with the usual metric $g = -f(r)\mathrm{d}t^2 + (f(r))^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_2^2$ in the coordinates $(t, r, \{\varphi\}_{S^{n-2}})$ and $f(r) = 1 - \dfrac{2M}{r}$. Find the conserved quantities along geodesics and prove that a radial free falling observer (any radial causal geodesic) reaches $r = 0$ in finite proper time (affine parameter λ).

Problem Sheet 6

Cosmology & Field Equations

Exercise 6.1. Let (M, g) be a $n+1$-dimensional Robertson-Walker spacetime.

(1) Compute the Ricci curvature of M using the result (from script) for Riemann curvature.

(2) Calculate the scalar curvature of M using the preceding result of Ricci curvature.

Exercise 6.2. The general form of the stress-energy tensor consistent with homogeneity and isotropy of a 4-dimensional Robertson-Walker spacetime (M, g) for a perfect fluid is given by

$$T = (\rho + p)\xi \otimes \xi + pg, \tag{6.1}$$

where ξ is a timelike (future-pointing) unit covector field on M whose dual is tangent to the flow lines of the fluid, ρ and p are the mass density and the pressure of the fluid considered on M. The radiation filled Robertson-Walker spacetime, per se, is characterised by $p(t) = \frac{\rho(t)}{3}$. Workout Einstein field equation on M for radiation using results from the preceding problem and describe how the "spatial scaling factor" (as described in the script) $f^2(t)$ behaves for sectional curvature $\kappa = -1, 0, +1$.

Exercise 6.3. Prove the following statement:

- A connected Riemannian or Lorentzian manifold which is isotropic at each point is necessarily homogeneous.

You can start by proving the following baby-case:

- A complete connected Riemannian manifold which is isotropic at each point is necessarily homogeneous.azasaz

Exercise 6.4. Let (M, g) and (N, h) be Lorentzian manifolds endowed with respective Levi-Civita connections. Show that

(1) A curve $I \ni s \mapsto \gamma(s) = (\alpha(s), \beta(s)) \in M \times N$ is a geodesic if and only if its projections α (resp. β) in M (resp. N) are both geodesics.

(2) $M \times N$ is (geodesically) complete if and only of both M and N are complete.

Problem Sheet 7

Time Orientation, Causality & Pregeodesics

Exercise 7.1. Let $M = \mathbb{R} \times S^1$. Give two explicit examples of Lorentz metrics on M such that the first one is time orientable and second one is not. Express them in the coframes $\mathrm{d}t$ and $\mathrm{d}\theta$ where t is the standard coordinate on $\mathbb{R}$ and θ the angular coordinate on S^1.

Moreover, prove that any manifold admitting a nonzero vector field can be endowed with a time-oriented Lorentzian metric.

Exercise 7.2. Let (M, g) be a Lorentzian manifold and for p, $q \in M$ set $I(p,q) := I^+(p) \cap I^-(q)$ and $J(p,q) := J^+(p) \cap J^-(q)$. Prove that the set of points at which the chronology [causality] condition fails is a (possibly empty) disjoint union of sets of the form $I(p,p)$ [$J(p,p)$].

Exercise 7.3. Prove that if X, Y are linearly independent vector fields on a smooth manifold N, there is a Lorentzian metric on N such that (separately) (a) X is timelike and Y is lightlike; (b) X and Y are both lightlike.

Give also examples of

- a Lorentzian metric on some open subset of $\mathbb{R}^3$ for which there are many closed timelike curves and many closed lightlike curves.
- a Lorentzian metric on $\mathbb{R} \times S^1$ for which some $J^+(p)$ is an open set with compact closure. (This last point is a bonus question open to discussion since we only have an unsatisfactory answer.)

Exercise 7.4. What is time-orientation? Give an example (except those in the script) of

(1) an oriented but non-time-oriented Lorentzian manifold.

(2) a time-oriented but non-oriented Lorentzian manifold.

(3) a non-oriented and non-time-oriented Lorentzian manifold.

(4) an oriented and time-oriented Lorentzian manifold.

Exercise 7.5. Which of the following Lorentzian manifolds are time-orientable and why? Describe their time-orientation when exist.

(1) Minkowski spacetime.

(2) $(M := \mathbb{S} \times \mathbb{S} \equiv \{(t,s) \mid |t| \leqslant 4\pi, 0 \leqslant x \leqslant 1\}$, $g := -\cos(t)\mathrm{d}t^2 + 2\sin(t)\mathrm{d}t\mathrm{d}x + \cos(t)\mathrm{d}x^2)$.

(3) de Sitter spacetime.

(4) Anti de Sitter spacetime.

(5) Robertson-Walker spacetime.

(6) Standard Schwarzschild spacetime.

(7) Kruskal-Szekeres extension of standard Schwarzschild spacetime.

(8) A simple-connected spacetime.

(9) $(\mathbb{R}^4$, $g := -\mathrm{d}t^2 + \mathrm{d}x^2 - \dfrac{1}{2}\mathrm{e}^{2\sqrt{2}cx}\mathrm{d}y^2 + \mathrm{d}z^2 - 2\mathrm{e}^{\sqrt{2}cx}\mathrm{d}t\mathrm{d}y)$ where (t, x, y, z) are standard coordinates on $\mathbb{R}^4$.

Exercise 7.6. Which of the following manifolds admit a Lorentzian metric? Give arguments in favour of your answer.

(1) Even d-dimensional sphere $\mathbb{S}^d$.

(2) Odd d-dimensional sphere $\mathbb{S}^d$.

(3) $d \in \mathbb{Z}$-dimensional torus.

Exercise 7.7. Is the time difference $\tau(p, q)$ between two points p and q in a time-oriented Lorentzian manifold that can be connected by a future-directed causal curve, symmetric (in p and q)?

Exercise 7.8. Prove that any simply connected 2-dimensional spacetime is strongly causal.

Exercise 7.9. Let (M, g) be a time-oriented Lorentzian manifold. Prove that the set of points at which M is strongly causal is open.

Exercise 7.10. If S is an achronal set of a time-oriented Lorentzian manifold then prove that its edge can be given by $\text{edge}(S) = \overline{S} \cap \overline{\partial I^+(S) \setminus S} = \overline{S} \cap \overline{\partial I^-(S) \setminus S}$.

Exercise 7.11. Prove that the set of points in (M, g), Lorentzian manifold, at which the chronology [causality] condition fails is a (possibly empty) disjoint union of sets of the form $I^+(p) \cap I^-(p)$ $[J^+(p) \cap J^-(p)]$.

Exercise 7.12. Let c be a regular ($\dot{c} \neq 0$) pregeodesic ($\nabla_{\dot{c}}\dot{c}(\lambda) = f(\lambda)\dot{c}(\lambda)$) in a Lorentzian manifold.

(1) Prove that $\gamma = c \circ \phi$ is a geodesic if and only if $\phi'' + (f \circ \phi)(\phi')^2 = 0$.

(2) If $g(\dot{c}, \dot{c})$ never vanishes, then any constant speed reparametrization of c is a geodesic.

(3) $g(\dot{c}, \dot{c})$ is always zero or never zero.

(4) If c is lightlike ($g(\dot{c}, \dot{c}) = 0$), w has a geodesic reparametrization.

Furthermore, prove that if w is a lightlike curve in a 2-dimensional manifold ($g(\dot{w}, \dot{w}) = 0$), w is pregeodesic.

Deduce that a curve c is a pregeodesic if and only if it can reparametrized as a geodesic.

Exercise 7.13. In a Lorentzian manifold (M, g), given a timelike unit vector U at a point $p \in M$ and an orthonormal basis $e_0, \dots, e_n$ such that $e_0 = U$, prove that

$$\text{Ric}(U, U) = -\sum_{i=1}^{n} K(e_i, U) =: -\overline{K}(U),$$

where $\overline{K}(U)$ is called average sectional curvature for the planes in the pencil of U.

Let now c be a unit timelike geodesic. Consider a Jacobi field J orthogonal to the curve c. J will be physically interpreted as the infinitesimal displacement between the particle moving along c and another particle moving along a nearby timelike geodesic. Then one can interpret $J'' := \nabla_{\dot{c}}\nabla_{\dot{c}}J$ as the relative tidal accerleration of the second particle as measured by the first.

(1) Prove that the radial component of J'' (the component in the direction of J) is equal to $K(\dot{c}, J)|J|$, where $|J| = \sqrt{g(J, J)}$ is the distance between the particles.

(2) Deduce that if the timelike plane spanned by $\dot{c}$ and J has positive sectional curvature, the particle will accelerate away from each other, while if the plane has negative sectional curvature, they will attract.

(3) Notice that $\text{Ric}(\dot{c}, \dot{c}) > 0$ corresponds to average attractive tidal forces.

Problem Sheet 8

Jacobi Fields & Conjugate Points

Exercise 8.1. Let (M, g) be a time-oriented Lorentzian manifold. Show that M satisfies the strong causality condition if and only if for any $p \in M$ the causal diamonds $J(q, q') := J^+(q) \cap J^-(q')$ with $q \ll p \ll q'$ form a neighborhood basis of p, i.e. for every open set U containing p there exist q, q' as above with $J(q, q') \subset U$.

Exercise 8.2. Argue whether the time separation function (also called time difference) τ on $H_1^n(r)$ (anti de Sitter spacetime) is finite or not. What happens on $\widetilde{H}_1^n(r)$, the universal cover of anti de Sitter spacetime?

Discuss whether $H_1^n(r)$ and $\widetilde{H}_1^n(r)$ satisfy the causality condition.

Exercise 8.3. Let M be a semi-Riemannian manifold, $p \in M$ and $\gamma : [0, b] \to M$ be a geodesic with $\gamma(0) = p$, $\gamma'(0) = v \in T_pM$. Then let $w \in T_pM$ with $w \neq 0$ and let J be a vector field along γ given by

$$J(t) = (\mathrm{d}\exp_p)_{tv}(tw).$$

Prove that J is a Jacobi field along γ and that the Taylor expansion of $|J(t)|^2 := g(J(t), J(t))$ for $t \to 0$ is given by

$$|J(t)|^2 = g(w, w)t^2 + \frac{1}{3}g(R(v, w)w, v)t^4 + o(t^4).$$

Use this to show that in normal coordinates (x^i) around p the metric components read for $|x| \to 0$

$$g_{ij}(x) = \varepsilon_i \delta_{ij} + \frac{1}{3}R_{kijl}x^k x^l + o(|x|^2),$$

where $\varepsilon_i = g_{ii}(0)$.

Exercise 8.4. Let M be a semi-Riemannian manifold and let X be a Killing vector field. Prove that the restriction of X to any geodesic c is a Jacobi field.

Exercise 8.5. Let M be a semi-Riemannian manifold, $p \in M$ and $\gamma : [0, b] \to M$ be a geodesic with $\gamma(0) = p$, $\gamma'(0) = v \in T_pM$. Then let $w \in T_v(T_pM)$ with $g(w, w) = 1$ and let J be a Jacobi field along γ given by

$$J(t) = (\mathrm{d}\exp_p)_{tv}(tw).$$

Prove that the Taylor expansion of $|J(t)|^2 := g(J(t), J(t))$ for $t \to 0$ is given by

$$|J(t)|^2 = t^2 - \frac{1}{3}g(R(v, w)v, w)t^4 + o(t^4).$$

Exercise 8.6. Let M be a semi-Riemannian manifold and let X be a Killing vector field. Prove that the restriction of X to any geodesic c is a Jacobi field. [Hint: Remember that $g(R(A, B)C, D) = g(R(C, D)A, B)$, for any vector field A, B, C, D.]

Exercise 8.7. Let (M, g) be a semi-Riemannian n-manifold. Show that for any vector field X on M it holds $\mathcal{L}_X \mu_g = (\operatorname{div} X)\mu_g$, where μ_g is the volume n-form built out of g.

Exercise 8.8. Consider Minkowski space with Cartesian coordinates $\{x^\alpha\}_{\alpha=0,\dots,n}$. Show that the translations ∂_α, $\alpha = 0, \dots, n$, the rotations $x^i\partial_j - x^j\partial_i$, $i = 1, \dots, n$, and the hyperbolic rotations $x^0\partial_i + x^i\partial_0$, $i = 1, \dots, n$ are all Killing fields.

Exercise 8.9. Assume here that $\dim M = n \geqslant 2$ and that (M, g) is a Riemannian manifold. Fix an orthonormal basis, and consider poly coordinates ξ on T_pM adapted to this basis, $\xi = r, \varphi^1, \dots, \varphi^{n-1}$, where $\varphi := \{\varphi^i\}$ are the standard coordinates on the sphere S^{n-1} and $r(X) = g(X, X)^{\frac{1}{2}}$. For instance, if $n = 2$ and $X \in T_pM$, $\xi(X) = (r, \phi)$ such that $X^1 = r\cos\phi$ and $X^2 = r\sin\phi$.

(1) Check from the definitions that the map $r \mapsto \exp_p(\xi^{-1}(r, \{\varphi\}))$ is a radial geodesic.

(2) Check that Gauss lemma translates into the requirements $g(\partial_r, \partial_r) = 1$, $g(\partial_r, \partial_{\varphi^i}) = 0$. In other words, the metric takes the local form

$$g = \mathrm{d}r^2 + g_{ij}\mathrm{d}\varphi^i \mathrm{d}\varphi^j.$$

(3) Explain what fails in the Lorentzian case.

Exercise 8.10. Prove the following: Suppose U is a unit vector field on a Riemannian manifold (M, g). Then $h = g - 2U^\flat \otimes U^\flat$ is a Lorentzian metric in which U timelike, so that (M, h) is time-orientable.

Problem Sheet 9

Exotic Metrics, Causal Pathologies & Stability

Exercise 9.1. Let (M, g) be a stably causal spacetime (i.e. there exists a global time function, also known as a function $t : M \to \mathbb{R}$ whose gradient is timelike) and h an arbitrary symmetric $(2,0)$-tensor field with compact support. Show that for sufficiently small $|\varepsilon|$ the tensor field $g_\varepsilon = g + \varepsilon h$ is still a Lorentzian metric on M, and (M, g_ε) is stably causal.

Exercise 9.2. In the following M is always a time-oriented Lorentzian manifold.

(1) For a subset $A \subseteq M$ show: $\operatorname{edge}(\overline{A}) \subseteq \operatorname{edge}(A)$ and $\operatorname{edge}(A) \setminus \operatorname{edge}(\overline{A}) \subseteq \overline{(\partial A) \setminus A}$. Give examples to show that both inclusions are proper inclusions in general.

(2) Give an example of some connected M and a closed, non-empty spacelike hypersurface of M that is not achronal.

(3) Give an example of a smooth hypersurface in $\mathbb{R}^2$ (Minkowski metric) that is acausal, but not spacelike.

Exercise 9.3. Prove that if X, Y are linearly independent vector fields on a smooth manifold N, there is a Lorentz metric on N such that (separately)

(1) X is timelike and Y is lightlike.

(2) X and Y are both lightlike.

Exercise 9.4. Give example of a Lorentzian metric on $\mathbb{R}^3$ for which there are many closed timelike curves and many closed lightlike curves. (Use the preceding exercise).

Exercise 9.5. Let M be the torus $T^2 = S^1 \times S^1$ endowed with the Lorentzian metric g by

$$g = \cos\theta(\mathrm{d}\psi^2 - \mathrm{d}\theta^2) + \sin\theta(\mathrm{d}\psi \otimes \mathrm{d}\theta + \mathrm{d}\theta \otimes \mathrm{d}\psi), \ (\theta, \psi) \in S^1 \times S^1.$$

Show that the closed defined by $\theta = \dfrac{\pi}{2}$ and $\theta = \dfrac{3\pi}{2}$ are lightlike geodesics such that the affine parameter λ satisfies $\dfrac{1}{2}\lambda\dfrac{\mathrm{d}\psi}{\mathrm{d}\lambda} = 1$. Conclude that if one goes through either of these curves infinitely many times in the negative ψ direction, only a finite amount of affine parameter is used up. (The manifold is lightlike geodesically incomplete).

Exercise 9.6. Suppose strong causality holds at each point of a compact set K in a spacetime M. If $\gamma : [0, b) \to M$ is a future inextendible causal curve that starts in K then eventually it leaves K and does not return, i.e., there exists $t_0 \in [0, b)$ such that $\gamma(t) \neq K$ for all $t \in [t_0, b)$.

Exercise 9.7. Let (M, g) be a stably causal spacetime (i.e. there exists a global time function, also known as a function $t : M \to \mathbb{R}$ whose gradient is timelike) and h an arbitrary symmetric $(2,0)$-tensor field with compact support. Show that for sufficiently small $|\varepsilon|$ the tensor field $g_\varepsilon = g + \varepsilon h$ is still a Lorentzian metric on M, and (M, g_ε) is stably causal.

Exercise 9.8. Let (M, g) be the quotient of the 2-dimensional Minkowski spacetime by the discrete group of isometries generated by the map $f(t, x) = (t + 1, x + 1)$. Show that (M, g) satisfies the chronology condition, but there exist arbitrarily small perturbations of (M, g) (in the sense of Exercise 9.7) which do not.

Exercise 9.9. Let (M, g) be the 2-dimensional spacetime obtained by removing the positive x-semi-axis of Minkowski 2-dimensional spacetime. Show that:

(1) (M, g) is stably causal but not globally hyperbolic.

(2) there exist points $p, q \in M$ such that $J^+(p) \cap J^-(q)$ is not compact.

(3) there exist points $p, q \in M$ with $q \in I^+(p)$ such that the supremum of the lengths of timelike curves connecting p to q is not attained by any timelike curve.

Problem Sheet 10

Cauchy Surfaces & Time Functions

Exercise 10.1. Prove that stable causality implies strong causality and show that the converse is not true by a counterexample.

Exercise 10.2. Which of the following manifolds admit a Cauthy hypersurface? Describe a Cauchy hypersurface of the respective spacetimes when exists.

(1) Minkowski spacetime.

(2) $(M := \mathbb{S} \times \mathbb{S} \equiv \{(t,s) \mid |t| \leqslant 4\pi, 0 \leqslant x \leqslant 1\}, \, g := -\cos(t)\mathrm{d}t^2 + 2\sin(t)\mathrm{d}t\mathrm{d}x + \cos(t)\mathrm{d}x^2)$.

(3) de Sitter spacetime.

(4) Anti de Sitter spacetime.

(5) Robertson-Walker spacetime.

(6) Standard Schwarzschild spacetime.

(7) Kruskal-Szekeres extension of standard Schwarzschild spacetime.

(8) A simple-connected spacetime.

(9) $(\mathbb{R}^4, \, g := -\mathrm{d}t^2 + \mathrm{d}x^2 - \dfrac{1}{2}\mathrm{e}^{2\sqrt{2}cx}\mathrm{d}y^2 + \mathrm{d}z^2 - 2\mathrm{e}^{\sqrt{2}cx}\mathrm{d}t\mathrm{d}y)$ where (t,x,y,z) are standard coordinates on $\mathbb{R}^4$.

Exercise 10.3. Which of the spacetimes in Exercise 10.2 admits a

- time function.
- temporal function.
- Cauchy time function.

Describe a these functions (time function, temporal function, and Cauchy time function) whenever they exist.

Exercise 10.4. Prove that the causal future/past $J^{\pm}(S)$ of a Cauchy hypersurface $S \subset M$ of a spacetime (M,g) are closed.

Exercise 10.5. Let (M,g) be a spacetime. Prove that the measure $\mathfrak{m}$ on M presented in the script satisfies the following properties.

(1) $\mathfrak{m}$ is finite: $\mathfrak{m}(M) < \infty$.

(2) $\mathfrak{m}$ is positive: $\mathfrak{m}(U) > 0$ for any non-empty open set $U \subset M$.

(3) $\mathfrak{m}$ satisfies $\mathfrak{m}(I^{\pm}(p)) = \mathfrak{m}(J^{\pm}(p))$ where $I^{\pm}$ and $J^{\pm}$ are the chronological and the causal future/past of any $p \in M$, respectively.

Exercise 10.6. Let (M,g) be the half-plane $M = \{(t,x) \in \mathbb{R}^2 \mid x > 0\}$ endowed with the metric $g = -x^2\mathrm{d}t^2 + \mathrm{d}x^2$.

(1) Show (M,g) is a Lorentzian manifold.

(2) Describe the lightlike geodesics passing through any point of the form $p = (0,x)$, $x > 0$, identifying (heuristically) the set $J^{\pm}(p)$.

(3) Show that (M, g) is stably causal (admits a time function, i.e. with timelike gradient).

(4) Show that (M, g) is isometric to (U, η), where η is the Minkowski metric and $U = \{(T, X) \in \mathbb{R}^2 \mid T < |X|, X > 0\}$. [Hint: Rindler coordinates.]

(5) Show (M, g) is globally hyperbolic. (At least two different arguments are possible.)

(6) Show that a static observer in (M, g) (for example $\gamma(\lambda) = (\lambda, 1)$) corresponds to constantly accelerating one in (U, g). (This tells us that this spacetimes is Minkowski as seen from a constantly accelerating observer, which hence has access only to the region U).

Exercise 10.7. Let (M, g) be a Lorentzian manifold $\mathcal{O} \subset M$ a globally hyperbolic subset. Can $\mathcal{O}$ be closed (with respect to the topology of M)? Discuss and provide examples if necessary.

Exercise 10.8. Let (M, g) be a globally hyperbolic Lorentzian manifold. Can M be closed?

Exercise 10.9. Let (M, g) be a time-oriented spacetime and $S \subset M$. Show that:

(1) $S \subset D_+(S)$.

(2) $D_+(S)$ is not necessarily open.

(3) $D_+(S)$ is not necessarily closed.

Problem Sheet 11

Global Hyperbolicity & Domains of Dependence

Exercise 11.1. Let (M, g) be the Schwarzschild spacetime with $M = \mathbb{R} \times (2m, \infty) \times S^2$ and

$$g(t, r, x) = -\left(1 - \frac{2m}{r}\right) \mathrm{d}t^2 + \frac{1}{1 - \dfrac{2m}{r}} \mathrm{d}r^2 + r^2 g_{S^2}(x)$$

and let $F : (2m, \infty) \to \mathbb{R}$ be a function whose derivative is $r \mapsto \left(1 - \dfrac{2m}{r}\right)^{-1}$.

(1) Show that F is strictly monotonously increasing with $\lim_{r \to (2m)} F(r) = -\infty$ and $\lim_{r \to \infty} F(r) = \infty$. Conclude that F has a continuous inverse $F^{-1} : \mathbb{R} \to (2m, \infty)$.

(2) Let now $\gamma : [a, b] \to M$, $\tau \mapsto (t(\tau), r(\tau), x(\tau))$ be a future-causal (piecewise $\mathcal{C}^1$) curve and $\tau_0 \in [a, b]$. Prove that

$$F_1(t(\tau)) \leqslant r(\tau) \leqslant F_2(t(\tau)) \text{ for all } \tau \geqslant \tau_0$$
$$F_2(t(\tau)) \leqslant r(\tau) \leqslant F_1(t(\tau)) \text{ for all } \tau \leqslant \tau_0$$

where $F_1(t) = F^{-1}(F(r(\tau_0)) + t(\tau_0) - t)$ and $F_2(t) = F^{-1}(F(r(\tau_0)) + t - t(\tau_0))$.

(3) Show that (M, g) is globally hyperbolic. [Hint: You may use without further proof that in the definition of globally hyperbolic the strong causality condition may be equivalently replaced by the (non-strong) causality condition.]

Exercise 11.2. Let (M, g) be the 2-dimensional spacetime obtained by removing the non-negative x-semi-axis of Minkowski 2-dimensional spacetime. Show that:

(1) (M, g) is stably causal but not globally hyperbolic.

(2) there exist points $p, q \in M$ with $q \in I^+(p)$ such that the supremum of the lengths of timelike curves connecting p to q is not attained by any timelike curve.

Exercise 11.3. Let (M, g) be the half-plane $M = \{(t, x) \in \mathbb{R}^2 \mid x > 0\}$ endowed with the metric $g = -x^2 \mathrm{d}t^2 + \mathrm{d}x^2$.

(1) Show (M, g) is a Lorentzian manifold.

(2) Describe the lightlike geodesics passing through any point of the form $p = (0, x)$, $x > 0$, identifying (heuristically) the sets $J^{\pm}(p)$.

(3) Show that (M, g) is stably causal (it is enough to show it admits a temporal function as in Exercise 9.1).

(4) Show that (M, g) is isometric to (U, η), where η is the Minkowski metric and $U = \{(T, X) \in \mathbb{R}^2 \mid T < |X|, X > 0\}$.

(5) Show (M, g) is globally hyperbolic.

Exercise 11.4. Prove that Cauchy hypersurfaces of a globally hyperbolic spacetime are diffeomorphic.

Exercise 11.5. Let (M, g) be a globally hyperbolic spacetime. Prove that $J^+(K) \cup J^-(L)$ is compact for any compact sets K, $L \subset M$.

Exercise 11.6. Let $(\mathbb{R}^2, g)$ be a time-oriented Lorentzian manifold and (t, x) standard coordinates on $\mathbb{R}^2$. Is it true that $(\mathbb{R}^2, g)$ is globally hyperbolic if the metric can be expressed in the form

$$g := -\mathrm{d}t^2 + f^2\mathrm{d}x^2$$

where f is a strictly positive function on $\mathbb{R}^2$, and for each $a \in \mathbb{R}$, $f(a, x) = 1$ for all $x \in \mathbb{R}$ outside a compact interval which depends on a? Reason in favour of your answer.

Exercise 11.7. Let M be a spacetime which is isometric to a smooth product manifold $\mathbb{I} \times S$ where $\mathbb{I} \subseteq \mathbb{R}$ is an interval and S is a Cauchy hypersurface. Which of the following statements is true? Argue in favour of your answer.

(1) M is globally hyperbolic if S is compact but $\mathbb{I}$ non-compact.

(2) M is globally hyperbolic if S is non-compact but $\mathbb{I}$ compact.

(3) M is globally hyperbolic if both S and $\mathbb{I}$ are non-compact.

(4) M is globally hyperbolic if both S and $\mathbb{I}$ compact.

Exercise 11.8. Prove that the time difference between any two points in a globally hyperbolic spacetime is always finite.

Exercise 11.9. Show that the chronology violating set of a compact spacetime is non-empty.

Exercise 11.10. Let $N = \mathbb{R}^2 \setminus \{0\}$ endowed with the metric

$$h = \frac{1}{u^2 + v^2}(\mathrm{d}u \otimes \mathrm{d}v + \mathrm{d}v \otimes \mathrm{d}u).$$

The group $\mathbb{Z}$ acts on N by isometries through

$$n \cdot (u, v) = (2^n u, 2^n v),$$

and this determines a metric g on $M = N/\mathbb{Z} \simeq T^2$. Show (M, g) is a Lorentzian manifold.

(1) Show that $u\partial_u + v\partial_v$ is a Killing vector field on N.

(2) Prove that if $\gamma(s) = (u(s), v(s))$ is a geodesic, then $\dfrac{u'v'}{r^2}$ and $\dfrac{uv' + vu'}{r^2}$ are constant, where $r^2 = u^2 + v^2$.

(3) Show that there exist lightlike incomplete geodesics (although M is compact).

Exercise 11.11. Let (M, g) be the 2-dimensional spacetime obtained by removing the positive x-semi-axis $(x \geqslant 0)$ of Minkowski 2-dimensional spacetime. Show that (M, g) is not reflecting.

Exercise 11.12. Let (M, g) be a time-oriented Lorentzian manifold. We say it is vicious at $p \in M$ if $I_+(p) \cap I_-(p) = M$. We say (M, g) is totally vicious if it is vicious for any $p \in M$. Prove that if (M, g) is vicious at p, (M, g) is totally vicious.

Exercise 11.13. Show that a globally hyperbolic spacetime is causally simple.

Problem Sheet 12

Singularities & Extensions

Exercise 12.1. A subset $\Omega \subset M$ in time-oriented Lorentzian manifold is called causally compatible if for all points $x \in \Omega$, $J_\pm^\Omega(x) = J_\pm^M(x) \cap \Omega$ holds. Let now $\Omega \subset M$ be a non-empty open subset of a time-oriented Lorentzian manifold M such that $J_+^M(p) \cap J_-^M(q)$ is contained in Ω for all $p, q \in \Omega$.

(1) Show that Ω is causally compatible.

(2) Show that if furthermore M is globally hyperbolic, then Ω is globally hyperbolic as well.

Exercise 12.2. Let (S, g_0) be a connected Riemannian manifold. Let $I \subset \mathbb{R}$ be an open interval and let $f : I \to \mathbb{R}$ be a smooth positive function. Let $M = I \times S$ and $g = -\mathrm{d}t^2 + f(t)^2 g_0$. We give M the time orientation with respect to which the vector field $\dfrac{\partial}{\partial t}$ is future directed.

Show that (M, g) is globally hyperbolic if and only if (S, g_0) is complete.

Exercise 12.3. Suppose that on a spacetime M of dimension n the timelike convergence condition holds true (i.e. at any point and for any timelike vector X, $\mathrm{Ric}(X, X) \geqslant 0$).

For a spacelike hypersurface, let v be the future directed timelike normal. Define $\theta := \mathrm{tr}(X \mapsto \nabla_X v)$. Let $s \mapsto \gamma(s)$ be a geodesic of the timelike congruence and suppose that $\theta_1 = \theta(s_1) < 0$.

Show that θ diverges to $-\infty$ within the domain $[s_1, s_1 + n/(-\theta_1)]$ provided the affine parameter extends sufficiently far.

Exercise 12.4. Which of the following Lorentzian manifolds is singular and why? (You can use results from the script and previous problem sheets).

(1) $(\mathbb{R}_+ \times \mathbb{R},\, g := -s^{-4}\mathrm{s} \otimes \mathrm{d}s + \mathrm{d}x \otimes \mathrm{d}x)$.

(2) $(\mathbb{R} \times \mathbb{R}_+,\, g := -x^2 \mathrm{d}t \otimes \mathrm{d}t + \mathrm{d}x \otimes \mathrm{d}x)$.

(3) Kruskal-Szekeres extension of Schwarzschild spacetime.

(4) Robertson-Walker spacetime.

(5) $(\mathbb{S} \times \mathbb{S},\, g = \cos(x)(\mathrm{d}y \otimes \mathrm{d}y - \mathrm{d}x \otimes \mathrm{d}x) + 2\sin(x)\mathrm{d}x \otimes_{\mathrm{symm}} \mathrm{d}y)$.

Describe the behaviour of Riemann curvature at the singularity (if exists).

Exercise 12.5. Let m be a negative parameter and

$$g_S := -\left(1 - \frac{2m}{r}\right)\mathrm{d}x^0 \otimes \mathrm{d}x^0 + \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r \otimes \mathrm{d}r + g_{\mathbb{S}^2},$$

a Lorentzian metric defined by the preceding expression, where $g_{\mathbb{S}^2}$ is the metric of the round unit 2-sphere $\mathbb{S}^2$. What is the maximal manifold on which g_S can be defined?

Exercise 12.6. Is the standard Schwarzschild spacetime $(M_S := \mathbb{R} \times (2m, \infty) \times \mathbb{S}^2, g_S)$ geodesically complete? Provide arguments in favour of your answer.

Exercise 12.7. Prove that the Einstein field equation without any matter is equivalent to $\mathrm{Ric} = 0$, where Ric is the Ricci curvature of the underlying Lorentzian manifold.

Exercise 12.8. Show that the (Fronsdal-)Kruskal-Szekeres extension

$$M := \{(t, x) \in \mathbb{R} \times \mathbb{R} \mid t^2 - x^2 < 1\} \times \mathbb{S}^2,$$
$$g := -32m^3 \frac{\mathrm{e}^{-\frac{r}{2m}}}{r}(-\mathrm{d}t \otimes \mathrm{d}t + \mathrm{d}x \otimes \mathrm{d}x) + r^2 g_{\mathbb{S}^2}$$

of the Schwarzschild spacetime, where r is implicitly defined by

$$t^2 - x^2 = \left(1 - \frac{r}{2m}\right) \mathrm{e}^{\frac{r}{2m}}$$

(1) Is a solution to the vacuum Einstein equation.

(2) Is (M, g) geodesically complete?

(3) Does the Schwarzschild event horizon exist in case $m < 0$? If so, what is the position of the horizon then?