

Lie Groups Problem Sheets

Dr. Christian Lange

Contents

0	Preliminaries — Matrix Lie Groups and Basic Examples	1
1	The Exponential Map and One-Parameter Subgroups	2
2	Lie Algebras and the Adjoint Representation	3
3	Lie Subgroups, Homomorphisms, and the Closed Subgroup Theorem	5
4	Lie Group Actions, Orbits, and Stabilizers	6
5	Representation Theory I — $\mathfrak{sl}_2(\mathbb{C})$ and Highest-Weight Vectors	7
6	Representation Theory II — Finite Dimensional $\mathfrak{sl}_2$ -Modules	8
7	Cartan Subalgebras, Roots, and Root Space Decomposition	9
8	Semisimple Lie Algebras and the Killing Form	10
9	Compact Lie Groups and Maximal Tori	11
10	Covering Groups — $SU(2)$ and $SO(3)$	12
11	Root Space Decomposition and Classification of 3D Compact Lie Groups	13
12	Haar Measure and the Peter–Weyl Theorem	14

Problem Sheet 0

Preliminaries — Matrix Lie Groups and Basic Examples

Exercise 0.1. Let $\mathcal{A}_1$ be the standard differentiable structure on $\mathbb{R}$, and $\mathcal{A}_2$ the one induced by the atlas $\{(\mathbb{R}, f(x) = x^3)\}$. Show that:

- (1) Show that $\mathcal{A}_1$ and $\mathcal{A}_2$ are not identical.
- (2) Are $(\mathbb{R}, \mathcal{A}_1)$ and $(\mathbb{R}, \mathcal{A}_2)$ diffeomorphic manifolds?

Exercise 0.2. Consider the curve $c : \mathbb{R} \rightarrow \mathbb{R}^2$. Examine whether c and the restrictions $c|_{(0,2\pi)}$ and $c|_{(0,\pi)}$ are

- (1) immersions,
- (2) injective immersions,
- (3) embeddings.

Exercise 0.3. Why is the embeddedness assumption in the following lemma.

Lemma. Consider the following commutative diagram of maps between manifolds, in which F is any differentiable map and G is an immersion.

$$\begin{array}{ccc} M & & \\ H \downarrow & \searrow F & \\ N & \xrightarrow{G} & P \end{array}$$

Then:

- (1) If G is an embedding, then H is continuous.
- (2) If H is continuous, then H is differentiable.

[Hint: Think of the preceding question.]

Exercise 0.4. Let $\varphi = (x_1, \dots, x_n) : U \rightarrow \mathbb{R}^n$ be a chart of a differentiable manifold M . Show that $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right] = 0$ for all i, j .

Exercise 0.5. Show that the vector field X defined by $X_{(a,b)} = b \frac{\partial}{\partial x} \Big|_{(a,b)} - a \frac{\partial}{\partial y} \Big|_{(a,b)}$ on $\mathbb{R}^2$ is complete and determine the corresponding one-parameter group of diffeomorphisms.

Exercise 0.6. Which definitions of the tangent space TM of a smooth manifold M are you familiar with? Can you show that for every $p \in M$ and $v \in T_p M$ there exists a smooth curve $c : (-\varepsilon, \varepsilon) \rightarrow M$, $\varepsilon > 0$, with $c(0) = p$ and $c'(0) = v$?

Problem Sheet 1

The Exponential Map and One-Parameter Subgroups

Exercise 1.1. Convince yourself that the symplectic group

$$Sp(2n, \mathbb{C}) = \{A \in M(2n, \mathbb{C}) \mid AJA^T = J\}$$

is indeed a closed matrix Lie group and show that its Lie algebra is given by

$$\mathfrak{sp}(2n) = \{X \in M(2n, \mathbb{C}) \mid XJ = -JX^T\}.$$

Exercise 1.2. Recall that the usual hermitian product on $M(n, \mathbb{C}) \cong \mathbb{C}^{n^2}$ and the usual Euclidean product on $M(n, \mathbb{R}) \cong \mathbb{R}^{n^2}$ can be expressed by the formulas $\langle X, Y \rangle = \text{tr}(XY^*)$ and $\langle X, Y \rangle = \text{tr}(XY^T)$, respectively.

Show that with respect to their subspace topology as closed matrix Lie groups $SO(n)$, $O(n)$, $SU(n)$ and $U(n)$ are compact, but $SL(n, \mathbb{R})$ and $SL(n, \mathbb{C})$ are not compact.

Exercise 1.3. Show that

- (1) $O(n)$ is not connected.
- (2) $SO(n)$, $U(n)$ and $SU(n)$ are connected.

Exercise 1.4. The Euclidean group $E(n)$ is the group of all maps $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $\|f(x) - f(y)\| = \|x - y\|$ for all $x, y \in \mathbb{R}^n$, where $\|\cdot\|$ is the standard Euclidean norm on $\mathbb{R}^n$. One can show that every $f \in E(n)$ with $f(0) = 0$ is a linear map. Show that:

- (1) The map $\varphi : \mathbb{R}^n \times O(n) \rightarrow E(n)$ defined by

$$\varphi(v, A)(x) = Ax + v,$$

is bijective.

- (2) The group $E(n)$ is isomorphic to a closed matrix Lie group.

Exercise 1.5. Convince yourself that the exponential map restricts to a well-defined map $\exp : \mathfrak{sl}(2, \mathbb{R}) \rightarrow SL(2, \mathbb{R})$. Show that:

- (1) $\exp(X) = \cosh \lambda \cdot I_2 + \frac{\sinh \lambda}{\lambda} \cdot X$, for all $X \in \mathfrak{sl}(2, \mathbb{R})$, $X \neq 0$, where $\lambda = \sqrt{-\det(X)}$ is an eigenvalue of X (in case of $\lambda = 0$ we define $\frac{\sinh \lambda}{\lambda} := 1$).
- (2) $\exp : \mathfrak{sl}(2, \mathbb{R}) \rightarrow SL(2, \mathbb{R})$ is not surjective.

[Hint: Show that matrices of the form $\begin{pmatrix} -1 & \pm 1 \\ 0 & -1 \end{pmatrix}$ are not contained in the image of $\exp$.]

Problem Sheet 2

Lie Algebras and the Adjoint Representation

Exercise 2.1. Recall that the product $M \times N$ of two smooth manifolds M and N has a natural product smooth structure. With respect to this smooth structure the projections $\pi_1 : M \times N \rightarrow M$ and $\pi_2 : M \times N \rightarrow N$ are smooth maps. Moreover, they induce isomorphisms $d\pi_1 \times d\pi_2 : T_{(p,q)}(M \times N) \rightarrow T_p M \times T_q N \cong T_p M \oplus T_q N$ for all $(p, q) \in M \times N$.

Let G be a Lie group. Denote the group multiplication and the group inversion by $\mu : G \times G \rightarrow G$ and $\tau : G \rightarrow G$, respectively.

- (1) Determine the differentials $d\mu_{(e,e)} : T_e G \oplus T_e G \rightarrow T_e G$ (under the above identification $T_{(e,e)}(G \times G) \cong T_e G \oplus T_e G$) and $d\tau_e : T_e G \rightarrow T_e G$.
- (2) For $X, Y \in T_e G$, let $\hat{X}, \hat{Y}$ be the left-invariant vector fields on G with $\hat{X}_e = X$ and $\hat{Y}_e = Y$, and let $\tilde{X}, \tilde{Y}$ be the right-invariant vector fields on G with $\tilde{X}_e = X$ and $\tilde{Y}_e = Y$. Show that $[\hat{X}, \hat{Y}]_e = -[\tilde{X}, \tilde{Y}]_e$.
- (3) Convince yourself that also right-invariant vector fields on G with the Lie bracket form a Lie algebra. Conclude that this Lie algebra is isomorphic to the Lie algebra of left-invariant vector fields on G with the Lie bracket.

Exercise 2.2. Let G and H be Lie groups and $\varphi : G \rightarrow H$ a group homomorphism which is smooth in a neighborhood of the neutral element e of G . Show that φ is smooth (and hence a Lie group homomorphism).

Exercise 2.3. Let $H \subset G$ be an abstract subgroup of a Lie group G . Show that its topological closure is a closed subgroup of G (and hence an embedded Lie subgroup by theorems mentioned in the script). Show that the closure is abelian if H is so.

Exercise 2.4. Let $T^2 = S^1 \times S^1$ be a 2-dimensional torus. Consider the Lie group homomorphism $\varphi : \mathbb{R} \rightarrow T^2$,
$$t \mapsto (e^{2\pi i t}, e^{2\pi i \alpha t})$$

where $\alpha \in \mathbb{R}$. Show that:

- (1) φ is an immersion.
- (2) If $\alpha \in \mathbb{Q}$, then φ is periodic and $G := \varphi(\mathbb{R})$ is a one-dimensional compact Lie subgroup of T^2 .
- (3) If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then there holds:
 - (a) φ is injective.
 - (b) G is dense in T^2 .
 - (c) G is not closed in T^2 .

Exercise 2.5. The skew-field of quaternions $\mathbb{H}$ is the set of matrices of type $\begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix}$ with $u, v \in \mathbb{C}$, endowed with the usual addition and multiplication of matrices. The fields $\mathbb{R}$ and $\mathbb{C}$ are embedded in $\mathbb{H}$ via $\mathbb{C} \ni u \mapsto \begin{pmatrix} u & 0 \\ 0 & \bar{u} \end{pmatrix}$. Set $\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $\mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. Then $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is a basis of the $\mathbb{R}$ -vector space $\mathbb{H}$, and one has

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

- (1) Show that $\mathbb{R}$ is the center of $\mathbb{H}$ with respect to multiplication.
- (2) Let $h = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{H}$ be a quaternion. Its real part and its conjugation are defined as $\text{Re}(h) := a$ and $\bar{h} := a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$, respectively. Show that conjugation is an anti-automorphism, i.e. $\bar{h}k = \bar{k}\bar{h}$ for all $h, k \in \mathbb{H}$. Show that we can define an inner product on $\mathbb{H}$ via $\langle h, k \rangle = \text{Re}(h\bar{k})$, which satisfies $\langle h, h \rangle = \det h$.
- (3) A quaternion h is called purely imaginary, if $\bar{h} = -h$. The subspace of purely imaginary quaternions is in a natural way identified with $\mathbb{R}^3$. Show that under this identification the cross product on $\mathbb{R}^3$ corresponds to the imaginary part of the quaternionic product.
- (4) Show that the unit quaternions $S^3 = \{h \in \mathbb{H} \mid \langle h, h \rangle = 1\}$ form a Lie group. Identify it as $SU(2)$.

Problem Sheet 3

Lie Subgroups, Homomorphisms, and the Closed Subgroup Theorem

Exercise 3.1. Show that the connected component of a Lie group G that contains the identity is a normal subgroup of G .

Exercise 3.2. Let G be a connected Lie group and let H be a Lie subgroup of finite index $|G : H|$. Show that $H = G$.

Exercise 3.3. The center of a Lie group G is the subgroup $Z(G) := \{g \in G \mid gh = hg \text{ for all } h \in G\}$.

- (1) Show that the center of $SO(n)$ is $\{\pm 1\}$ if $n > 2$ is even and $\{1\}$ if n is odd.
- (2) Determine the center of $U(n)$ and $SU(n)$.

Exercise 3.4. Show that $U(n)$ and $S^1 \times SU(n)$ are diffeomorphic, but not isomorphic as Lie groups.

Exercise 3.5. Let $\mathfrak{g}$ be a 2-dimensional Lie algebra over $\mathbb{R}$ or $\mathbb{C}$. Show that $\mathfrak{g}$ is either abelian or there exists a basis $\{X, Y\}$ of $\mathfrak{g}$ with $[X, Y] = Y$. Deduce that up to isomorphism there are only two 2-dimensional Lie algebras.

Can you realize the non-abelian 2-dimensional Lie algebra over $\mathbb{R}$ as a Lie subalgebra of $\mathfrak{gl}(2, \mathbb{R}) = M(2, \mathbb{R})$? What is the corresponding connected Lie subgroup of $GL(2, \mathbb{R})$?

Problem Sheet 4

Lie Group Actions, Orbits, and Stabilizers

Exercise 4.1. Let $\varphi : G \rightarrow H$ be a Lie group homomorphism between connected Lie groups which is also a covering. Show that the kernel of φ is contained in the center of G . Conclude that the fundamental group of a Lie group is abelian.

Exercise 4.2. Recall the statements of Exercise 2.5. Now consider the Lie group of unit-quaternions $S^3 = \{h \in \mathbb{H} \mid \langle h, h \rangle = 1\}$. The adjoint representation $\text{Ad} : S^3 \rightarrow GL(3, \mathbb{R})$ of S^3 is given by $h \mapsto \text{Ad}_h$, where $\text{Ad}_h : x \in \mathbb{R}^3 \mapsto hx\bar{h} \in \mathbb{R}^3$, and $\mathbb{R}^3$ is identified with the imaginary quaternions. Show that:

(1) $\text{Ad} : S^3 \rightarrow SO(3)$ is a Lie group homomorphism. Determine its kernel.

(2) $\text{Ad} : S^3 \rightarrow SO(3)$ is a covering map.

[Hint: By chapter 4 of script, one only needs to check that the differential at the identity is an isomorphism.]

Exercise 4.3. Consider the map

$$\begin{aligned} \varphi : S^3 \times S^3 &\rightarrow SO(4) \\ (g, h) &\mapsto \varphi(g, h), \end{aligned}$$

where $\varphi(g, h) : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ is the linear map $\varphi(g, h)(x) = gx\bar{h}$ under the identification $\mathbb{R}^4 \cong \mathbb{H}$. Show that φ is a well-defined Lie group homomorphism and a covering. What can you say about the possible Lie groups with Lie algebra $\mathfrak{so}(4)$ up to isomorphism?

Exercise 4.4. Use the Gram-Schmidt process to show that $SL(2, \mathbb{C})$ deformation retracts onto $SU(2)$. See the following for a related and more general version of this statement.

Theorem. Let $G \subset GL(n, \mathbb{C})$ be a closed matrix Lie group which is defined as the common zero locus of some set of real-valued polynomials in the real and imaginary parts of the matrix entries, and is closed under taking the adjoint matrix. Let $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{C})$ be its Lie algebra, and $K := G \cup U(n)$. Then the map

$$K \times \mathfrak{p} \rightarrow G(k, X) \mapsto ke^X$$

is a homeomorphism, where $\mathfrak{p}$ is the subspace of hermitian matrices in $\mathfrak{g}$.

In particular, this shows that $SL(2, \mathbb{C})$ is simply connected. Have a look at the following proposition.

Proposition. The universal cover $\tilde{G}$ of $SL(2, \mathbb{R})$ can not be realized as a matrix group: there does not exist an injective Lie group homomorphism $\varphi : \tilde{G} \rightarrow GL(n, \mathbb{R})$.

Which uses this fact to prove that there does not exist an injective Lie group homomorphism from the universal covering Lie group of $SL(2, \mathbb{R})$ to some $GL(n, \mathbb{R})$. Convince yourself that the two homomorphisms appearing in the proof have indeed the same differential.

Problem Sheet 5

Representation Theory I — $\mathfrak{sl}_2(\mathbb{C})$ and Highest-Weight Vectors

Exercise 5.1. Let G be a Lie group. Show that there exists a neighborhood of the neutral element e in G , which contains no subgroups except $\{e\}$.

Exercise 5.2. Determine all one-parameter groups $\varphi : (\mathbb{R}, +) \rightarrow (\mathbb{R}^*, \cdot)$.

Exercise 5.3. Let G be a connected Lie group. Show that $\exp : \mathfrak{g} \rightarrow G$ is a group homomorphism if and only if G is abelian. (Here we consider $\mathfrak{g}$ as a Lie group with respect to addition).

Exercise 5.4. Show that the exponential maps of T^n , $SO(n)$ and $SU(n)$ are surjective.

Problem Sheet 6

Representation Theory II — Finite Dimensional $\mathfrak{sl}_2$ -Modules

Exercise 6.1. Let $\varphi : G \rightarrow H$ be a Lie group homomorphism. Show that $\ker \varphi$ is a closed Lie subgroup of G with Lie algebra $\ker d\varphi$.

Exercise 6.2.

- (1) Let G be an abelian Lie group. Show that the inversion $i : G \rightarrow G$ is a Lie group homomorphism with differential $di : \mathfrak{g} \rightarrow \mathfrak{g}$. Conclude that $\mathfrak{g}$ is abelian.

$$a \mapsto a^{-1}$$

$$X \mapsto -X$$

- (2) Let G be a connected Lie group with abelian Lie algebra $\mathfrak{g}$. Use the adjoint representation to show that G is abelian.

Exercise 6.3. Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{K}$. A derivation of $\mathfrak{g}$ is a $\mathbb{K}$ -linear map $D : \mathfrak{g} \rightarrow \mathfrak{g}$ such that $D[X, Y] = [DX, Y] + [X, DY]$ for all $X, Y \in \mathfrak{g}$. Show that

- (1) The set $\text{Der}(\mathfrak{g})$ of all derivations of $\mathfrak{g}$ is a subalgebra of $\text{End}(\mathfrak{g})$, where $\text{End}(\mathfrak{g})$ is endowed with the natural Lie commutator bracket.

- (2) $\text{ad} : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{g})$, where $\text{ad}_X(Y) := [X, Y]$, is well-defined Lie algebra homomorphism.

$$X \mapsto \text{ad}_X$$

- (3) Show that the image of ad in $\text{Der}(\mathfrak{g})$ is an ideal.

Exercise 6.4. Let G be a connected Lie group. Show that the center Z of G coincides with the kernel of the adjoint representation of G . Moreover, show that its Lie algebra is given by $\{X \in \mathfrak{g} \mid [X, Y] = 0 \text{ for all } Y \in \mathfrak{g}\} = \ker(\text{ad})$.

Problem Sheet 7

Cartan Subalgebras, Roots, and Root Space Decomposition

Exercise 7.1. The group $\text{Aut}(\mathfrak{g})$ of Lie algebra automorphisms (i.e. linear isomorphisms $A : \mathfrak{g} \rightarrow \mathfrak{g}$ which preserve that Lie bracket in the sense that $A([X, Y]) = [A(X), A(Y)]$) of a finite dimensional real Lie algebra $\mathfrak{g}$ is a closed subgroup of $GL(\mathfrak{g})$ and hence a Lie subgroup. Show that its Lie algebra is given by the algebra $\text{Der}(\mathfrak{g})$ of derivations of $\mathfrak{g}$.

Exercise 7.2. Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras over a field K and let $\pi : \mathfrak{h} \rightarrow \text{Der}(\mathfrak{g})$ be a Lie algebra homomorphism. Show that there exists a unique Lie algebra structure on the vector space $\mathfrak{g} \oplus \mathfrak{h}$ such that $\mathfrak{g} \subset \mathfrak{g} \oplus \mathfrak{h}$ is an ideal, $\mathfrak{h} \subset \mathfrak{g} \oplus \mathfrak{h}$ is a Lie subalgebra, the restriction of the Lie bracket of $\mathfrak{g} \oplus \mathfrak{h}$ to $\mathfrak{g}$ and $\mathfrak{h}$ respectively coincides with their initial Lie brackets, and such that $[Y, X] = \pi(Y)(X)$ for all $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$. This Lie algebra is called the semidirect sum of $\mathfrak{g}$ and $\mathfrak{h}$, and is denoted as $\mathfrak{g} \oplus_{\pi} \mathfrak{h}$.

Exercise 7.3. Let G and H be Lie groups and $\tau : H \rightarrow \text{Aut}(G)$ a Lie group homomorphism such that the map

$$\begin{aligned} H \times G &\rightarrow G \\ (h, g) &\mapsto \tau(h)(g) \end{aligned}$$

is smooth. Show that

- (1) The semidirect product $G \rtimes_{\tau} H$ of G and H with respect to τ is a Lie group.
- (2) G and H are Lie subgroups of $G \rtimes_{\tau} H$, and G is moreover a normal subgroup.
- (3) The Lie algebra of $G \rtimes_{\tau} H$ is given by $\mathfrak{g} \oplus_{d\tau} \mathfrak{h}$. Here $d\tau : \mathfrak{h} \rightarrow \text{Der}(\mathfrak{g})$ is the differential of the Lie group homomorphism $H \rightarrow \text{Aut}(G)$. Notice that, by Exercise 7.1, the Lie algebra of $\text{Aut}(G)$ is Lie algebra of

$$\begin{aligned} h &\mapsto d(\tau(h)) \\ \text{Aut}(\mathfrak{g}) &\text{ is given by } \text{Der}(\mathfrak{g}). \end{aligned}$$

Exercise 7.4. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{R})$. Consider the abelian Lie algebra $\mathbb{R}^2$ with a fixed basis $\{X, Y\}$, and the Lie algebra $\mathbb{R}$ with a fixed basis $\{Z\}$. We define a Lie algebra homomorphism $\pi_A : \mathbb{R} \rightarrow \text{Der}(\mathbb{R}^2)$ by $\pi_A(Z)(X) = aX + cY$ and $\pi_A(Z)(Y) = bX + dY$.

- (1) Show that the semidirect product $\mathbb{R}^2 \oplus_{\pi_A} \mathbb{R}$ can be realized as a matrix Lie algebra, by identifying X with $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, Y with $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, and Z with $\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 0 \end{pmatrix}$.
- (2) Show that $\mathbb{R}^2 \oplus_{\pi_A} \mathbb{R}$ is a solvable Lie algebra which is not nilpotent.
- (3) Show that the Lie algebras $\mathbb{R}^2 \oplus_{\pi_A} \mathbb{R}$ with $A = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, $a > 1$, are not isomorphic to each other.

Hence, there are uncountable many pairwise non isomorphic real solvable 3-dimensional Lie algebras.

Problem Sheet 8

Semisimple Lie Algebras and the Killing Form

Exercise 8.1. Let $\mathfrak{g}$ be a semisimple Lie algebra. Show that the only ideals in $\mathfrak{g}$ are the sums of some of its simple ideals.

Exercise 8.2. Prove that the Killing form of a nilpotent Lie algebra is zero.

Exercise 8.3. Show that $\mathfrak{sl}(n, \mathbb{C})$ is simple for all $n > 1$. To that end, define $E_{ij} \in M(n, \mathbb{C})$ as the matrix with all entries zero except a single entry 1 in the i th row and j th column, and check the following identity

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{kj},$$

where $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ is the Kronecker- δ . For a nontrivial ideal $\mathfrak{h} \subset \mathfrak{sl}(n, \mathbb{C})$ then proceed as follows:

- (1) Show that there exists a matrix $A = (a_{ij}) \in \mathfrak{h}$ such that $a_{kl} \neq 0$ for some $k \neq l$.
- (2) Show that this implies $E_{lk} \in \mathfrak{h}$.
- (3) Deduce that $\mathfrak{h} = \mathfrak{sl}(n, \mathbb{C})$.

Exercise 8.4. Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra.

- (1) Look up the statement of Schur's lemma.
- (2) Use Schur's lemma to show that if $\mathfrak{g}$ is simple and semisimple, then every ad-invariant symmetric bilinear form C on $\mathfrak{g}$ is a scalar multiple of the Killing form. Here, a symmetric bilinear form C on $\mathfrak{g}$ is called ad-invariant, if $C(\text{ad}_X Y, Z) + C(Y, \text{ad}_X Z) = 0$.

[Hint: Define $A : \mathfrak{g} \rightarrow \mathfrak{g}$ by $B(A(X), Y) = C(X, Y)$.]

- (3) Conclude that the Killing form B of $\mathfrak{sl}(n, \mathbb{C})$, $n \geq 2$, satisfies

$$B(X, Y) = 2n \text{tr}(X, Y),$$

for all $X, Y \in \mathfrak{sl}(n, \mathbb{C})$.

Problem Sheet 9

Compact Lie Groups and Maximal Tori

Exercise 9.1. Prove that a real Lie algebra with a positive-definite Killing form must be zero-dimensional.

Exercise 9.2. Explain why $\mathfrak{sl}(2, \mathbb{R})$ is not a compact Lie algebra.

Exercise 9.3. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Show that an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ is Ad-invariant if and only if

$$\langle \operatorname{ad}_X Y, Z \rangle + \langle Y, \operatorname{ad}_X Z \rangle = 0$$

holds for all $X, Y, Z \in \mathfrak{g}$ (in which case we say it is ad-invariant).

Exercise 9.4. Let $\mathfrak{g}$ be a complex Lie algebra. By restricting the scalars, we can view it as a real Lie algebra $\mathfrak{g}^{\mathbb{R}}$. Show that $\mathfrak{g}$ must be abelian, if $\mathfrak{g}^{\mathbb{R}}$ is a compact Lie algebra.

Problem Sheet 10

Covering Groups — $SU(2)$ and $SO(3)$

Exercise 10.1. Let $\mathfrak{g}$ be a complex Lie algebra. Show that $\mathfrak{g}$ is semisimple (i.e. that its Killing form is nondegenerate) if and only if its realification $\mathfrak{g}^{\mathbb{R}}$ is semisimple.

Exercise 10.2. Let $\mathfrak{g}$ be a real Lie algebra. Show that $\mathfrak{g}$ is semisimple if and only if its complexification $\mathfrak{g}^{\mathbb{C}}$ is semisimple. Show that $\mathfrak{g}^{\mathbb{C}}$ is simple, if $\mathfrak{g}$ is a compact simple Lie algebra.

Exercise 10.3. Let $p, q \geq 1$ be positive integers with $n = p + q$. One can show that

$$SU(p, q) = \{A \in M(n, \mathbb{C}) \mid A^* I_{p, q} A = I_{p, q}, \det A = 1\},$$

where $I_{p, q} = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}$, is a Lie group with Lie algebra

$$\mathfrak{su}(p, q) = \left\{ X = \begin{pmatrix} X_1 & X_2 \\ X_2^* & X_3 \end{pmatrix} \in M(n, \mathbb{C}) \mid X_1^* = -X_1, X_3^* = -X_3, \operatorname{tr}(X) = 0 \right\}.$$

(1) Show that the complexification of $\mathfrak{su}(p, q)$ is isomorphic to

$$\mathfrak{sl}(n, \mathbb{C}) = \{X \in M(n, \mathbb{C}) \mid \operatorname{tr}(X) = 0\}.$$

(2) For which p and q is $\mathfrak{su}(p, q)$ a compact real form of $\mathfrak{sl}(n, \mathbb{C})$?

(3) Conclude that the Lie groups $SU(2)$ ($n \geq 2$), $SU(p, q)$ ($p, q \geq 1$) and $SL(n, \mathbb{R})$ ($n \geq 2$) are simple.

Exercise 10.4. Let $\mathfrak{g} \subset \mathfrak{sl}(n, \mathbb{C})$ be a Lie subalgebra that is simple and invariant under taking the Hermitian conjugate, i.e. $X^* \in \mathfrak{g}$ if $X \in \mathfrak{g}$. Show that $\mathfrak{g}$ has a compact real form.

Problem Sheet 11

Root Space Decomposition and Classification of 3D Compact Lie Groups

Exercise 11.1. Show that the diagonal matrices in $SU(n)$ and $U(n)$, respectively, form a maximal torus. Conclude that the diagonal matrices in $\mathfrak{sl}(n, \mathbb{C})$ form a Cartan subalgebra.

Exercise 11.2. Consider the Lie algebra $\mathfrak{sl}(n, \mathbb{C})$ with the diagonal matrices as a Cartan subalgebra $\mathfrak{h}$. Let E_{ij} be the matrix with a 1 in the i, j -entry and zeros elsewhere. Let $e_i \in \mathfrak{h}^*$ be the linear functional defined by $e_i(\text{diag}(d_1, \dots, d_n)) = d_i$. Recall the relation $[E_{kl}, E_{ij}] = \delta_{li}E_{kj} - \delta_{kj}E_{il}$. Determine the roots and the root space decomposition of $\mathfrak{sl}(n, \mathbb{C})$ with respect to the Cartan subalgebra $\mathfrak{h}$.

Exercise 11.3. In this exercise we want to prove the uniqueness part in the following statement about representations of $\mathfrak{sl}(2, \mathbb{C})$: For each integer $n \geq 1$ there exists, up to isomorphism, a unique irreducible representation $\pi : \mathfrak{sl}(2, \mathbb{C}) \rightarrow \text{End}(V)$ on a complex vector space of dimension n . It is uniquely determined by the property that there exists a basis $\{v_0, \dots, v_{n-1}\}$ of V such that

- (1) $\pi(h)(v_i) = (n - 1 - 2i)v_i$
- (2) $\pi(f)(v_i) = v_{i+1}$ for $i < n - 1$
- (3) $\pi(f)(v_{n-1}) = 0$
- (4) $\pi(e)(v_0) = 0$
- (5) $\pi(e)(v_i) = i(n - 1)v_{i-1}$ for $i \geq 1$.

Here are some hints: Start with any eigenvector v of $\pi(h)$ and through applications of $\pi(e)$ find an eigenvector v_0 of $\pi(h)$ such that $\pi(e)v_0 = 0$. Set $v_i = \pi(f)^i v_0$ and show that there is a minimal k such that $\pi(f)^{k+1} v_0 = 0$. Prove that all desired conditions are satisfied and conclude that $k = n - 1$.

Remark. To show the existence part one can just define a representation of $\mathfrak{sl}(2, \mathbb{C})$ by the above properties and check that all desired conditions are fulfilled. The resulting representations can for instance also be described in terms of bivariate homogenous polynomials. See the following theorem:

Theorem. φ_k is irreducible and $\hat{S}^3 = \{\varphi_k\}_{k=0}^\infty$.

Exercise 11.4. Use the root space decomposition to:

- (1) Classify compact connected Lie groups of dimension 3.
- (2) Prove that there exist no compact semisimple Lie groups in dimensions 4, 5 and 7.

Problem Sheet 12

Haar Measure and Peter–Weyl Theorem

Exercise 12.1. Show that the Weyl group of $\mathfrak{sl}(n, \mathbb{C})$ is isomorphic to the symmetric group S_n .

Exercise 12.2. Consider the Lie algebra $\mathfrak{so}(2n, \mathbb{C})$, $n \geq 2$, and define

$$\mathfrak{h} = \left\{ H = \begin{pmatrix} 0 & ih_1 & & & \\ -ih_1 & 0 & & & \\ & & 0 & ih_2 & \\ & & -ih_2 & 0 & \\ & & & \ddots & \\ & & & & 0 & ih_n \\ & & & & -ih_n & 0 \end{pmatrix} \mid h_i \in \mathbb{C} \right\} \subset \mathfrak{so}(2n, \mathbb{C}).$$

Let $e_i \in \mathfrak{h}^*$, $1 \leq i \leq n$, be the linear map that maps the matrix H to h_i .

- (1) Show that $\text{ad}_H E_\alpha = \alpha(H)E_\alpha$ for all $H \in \mathfrak{h}$ and $\alpha = \pm e_i \pm e_j$, where the matrix E_α is nontrivial at most in the i, j -blocks, where it is given by

$$E_\alpha = \begin{pmatrix} 0 & X_\alpha \\ -X_\alpha^T & 0 \end{pmatrix}.$$

Here

$$\begin{aligned} X_{e_i - e_j} &= \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, & X_{e_i + e_j} &= \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \\ X_{-e_i + e_j} &= \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, & X_{-e_i - e_j} &= \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}. \end{aligned}$$

- (2) Show that

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{so}(2n, \mathbb{C}) \mid [H, X] = \alpha(H)X, \forall H \in \mathfrak{h}\} = \mathbb{C}E_\alpha,$$

for $\alpha = \pm e_i \pm e_j$ and that

$$\mathfrak{so}(2n, \mathbb{C}) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

where $\Delta := \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\}$. Conclude that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{so}(2n, \mathbb{C})$ and that the decomposition above is the root space decomposition of $\mathfrak{so}(2n, \mathbb{C})$ with respect to $\mathfrak{h}$.

Exercise 12.3. Let $\mathfrak{a} \subset \mathfrak{so}(2n, \mathbb{C})$ be an ideal in $\mathfrak{so}(2n, \mathbb{C})$, $\mathfrak{a} \neq 0$, $n \geq 3$. Using the notation from the previous exercise, show that

- (1) $\mathfrak{a} \not\subset \mathfrak{h}$.

- (2) There is some $\alpha \in \Delta$ such that $\mathfrak{g}_\alpha \subset \mathfrak{a}$.

[Hint: Consider $X \in \mathfrak{a}$, $X \notin \mathfrak{h}$. Let $\alpha_0 \in \Delta$ be such that the component of X in $\mathfrak{g}_{\alpha_0}$ is nontrivial. Successively apply Lie brackets such that a new X has no component in $\mathfrak{g}_\alpha$ for any $\alpha \neq \alpha_0$.]

- (3) $\mathfrak{g}_\beta \subset \mathfrak{a}$, for all $\beta \in \Delta \cup \{0\}$. Conclude that $\mathfrak{so}(2n, \mathbb{C})$ is simple.

Exercise 12.4. Choose a set of simple roots for $\mathfrak{so}(2n, \mathbb{C})$, $n \geq 3$, and use them to determine the Dynkin diagram of $\mathfrak{so}(2n, \mathbb{C})$.

[Hint: Recall that the Killing form of $\mathfrak{so}(2n, \mathbb{C})$ needs to be proportional to $\text{tr}(XY)$ by Exercise 8.4, since $\mathfrak{so}(2n, \mathbb{C})$ is simple by Exercise 12.3, semisimple because we have a compact real form, and the trace is an ad-invariant bilinear form.]