Lie Algebras Problem Sheets

Dr. Joachim Wehler

Contents

Problem Sheet 1

Fundamentals

Exercise 1.1. Determine the radius of convergence of the following power series:

(1) $\sum\limits_{v=0}^{\infty} \frac{1}{v!} \cdot z^v$

(2) $\sum\limits_{v=0}^{\infty} z^v$

(3) $\sum\limits_{v=1}^{\infty} \frac{(-1)^{v+1}}{v} \cdot z^v$

(4) $\sum\limits_{v=0}^{\infty} \frac{(-1)^v}{(2v)!} \cdot z^{2v}$

(5) $\sum\limits_{v=0}^{\infty} v! \cdot z^v$

(6) Which well-known functions do the power series (1)-(5) represent?

Exercise 1.2. Consider the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Prove that the Euclidean norm

$$\|x\| := \sqrt{\sum_{i=1}^{n} |x_i|^2}, \quad x = (x_1, \ldots, x_n) \in \mathbb{K}^n,$$

is a norm on the vector space $\mathbb{K}^n$, i.e. it satisfies

(1) $\|x\| = 0 \Leftrightarrow x = 0$, $x \in \mathbb{K}^n$,

(2) $\|\lambda \cdot x\| = |\lambda| \cdot \|x\|$, $\lambda \in \mathbb{K}$, $x \in \mathbb{K}^n$,

(3) Triangle inequality: $\|x + y\| \leqslant \|x\| + \|y\|$, $x, y \in \mathbb{K}^n$

Exercise 1.3. Consider the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Show that for matrices $A, B \in M(n \times n, \mathbb{K})$ the operator norm

$$\|A\| := \sup\{\|Ax\| : x \in \mathbb{K}^n \text{ and } \|x\| \leqslant 1\}$$

is a norm on the vector space $M(n \times n, \mathbb{K})$, i.e. it satisfies

(1) $\|A\| = 0 \Leftrightarrow A = 0$,

(2) $\|\lambda \cdot A\| = |\lambda| \cdot \|A\|$, $\lambda \in \mathbb{K}$,

(3) Triangle inequality: $\|A + B\| \leqslant \|A\| + \|B\|$
In addition show

(4) $\|A \cdot B\| \leqslant \|A\| \cdot \|B\|$,

(5) $\|\mathbb{1}\| = 1$ with the unit matrix $\mathbb{1} \in M(n \times n, \mathbb{K})$.

Exercise 1.4. Consider the matrix

$$A := \begin{pmatrix} 3 & 4 & 3 \\ -1 & 0 & -1 \\ 1 & 2 & 3 \end{pmatrix} \in M(3 \times 3, \mathbb{C}).$$

(1) Determine the characteristic polynomial of A.

(2) Determine the eigenvalues and eigenspaces of A.

(3) Is A diagonalizable?

Problem Sheet 2

Matrix Series and the Exponential Map

Exercise 2.1. The complex geometric series

$$\sum_{v=0}^{\infty} z^v$$

has radius of convergence $R = 1$. Hence the series

$$\sum_{v=0}^{\infty} A^v \in M(n \times n, \mathbb{C})$$

is well-defined for any matrix $A \in M(n \times n, \mathbb{C})$ with $\|A\| < 1$.

Show: The matrix $\mathbb{1} - A \in M(n \times n, \mathbb{C})$ is invertible with

$$(\mathbb{1} - A)^{-1} = \sum_{v=0}^{\infty} A^v.$$

[Hint: Imitate the proof of the analogous result for the complex geometric series.]

Exercise 2.2. The complex logarithmic series

$$\log(1+z) = \sum_{v=1}^{\infty} \frac{(-1)^{v+1}}{v} \cdot z^v$$

has radius of convergence $R = 1$. Hence the series

$$\log(\mathbb{1} + A) := \sum_{v=1}^{\infty} (-1)^{v+1} \cdot \frac{A^v}{v} \in M(n \times n, \mathbb{C})$$

is well-defined for any matrix $A \in M(n \times n, \mathbb{C})$ with $\|A\| < 1$. Consider an open subset $I \subset \mathbb{R}$ and a differentiable function

$$B : I \to M(n \times n, \mathbb{C})$$

with $\|B(t) - \mathbb{1}\| < 1$ and $[B'(t), B(t)] = 0$ for all $t \in I$.

Show: For all $t \in I$ the inverse $B(t)^{-1}$ exists and

$$\frac{\mathrm{d}}{\mathrm{d}t} \log B(t) = B(t)^{-1} \cdot B'(t) = B'(t) \cdot B(t)^{-1}.$$

[Hint: In order to compute $B(t)^{-1}$ apply Exercise 2.1 with $A := \mathbb{1} - B(t)$.]

Exercise 2.3. Consider the endomorphism $f \in \mathrm{End}(\mathbb{C}^2)$ defined with respect to the canonical basis by the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \in M(2 \times 2, \mathbb{C}).$$

(1) Show that

$$A_s := \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \qquad \text{(semisimple)}$$

and

$$A_n := \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \qquad \text{(nilpotent)}$$

are not the matrices of the Jordan decomposition of f.

(2) Compute the matrices of the Jordan decomposition of f.

Exercise 2.4. Provide the group

$$GL(n, \mathbb{K}) \subset \mathbb{K}^{n^2}$$

with the induced topology from the Euclidean space.

Show: Each open subgroup

$$H \subset GL(n, \mathbb{K})$$

is also closed.

[Hint: You may use that a subspace is closed if and only if its complement is open.]

Problem Sheet 3

Jordan Decomposition and Basic Structure

Exercise 3.1. Determine a matrix $A \in \mathfrak{gl}(2, \mathbb{C})$ with

$$\exp A = \begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}, \ b \in \mathbb{R}^*.$$

Exercise 3.2. Consider a finite-dimensional complex vector space V and an endomorphism $f \in \mathrm{End} V$. Show:

(1) If f is diagonalizable then f is semisimple.

(2) If f is semisimple then f is diagonalizable.
[Hint: You may use the decomposition $V = \bigoplus_\lambda V^\lambda(f)$ and prove:

$$p_{\min}(T) = \prod_\lambda (T - \lambda) \Rightarrow V^\lambda(f) \subset V_\lambda(f).]$$

(3) The sum of two semisimple, commuting endomorphisms of V is semisimple.

Exercise 3.3. Consider a finite-dimensional $\mathbb{K}$-vector space V. Show: The sum of two nilpotent, commuting endomorphisms of V is nilpotent.

Exercise 3.4. Show that the subgroup of invertible matrices with rational entries

$$GL(2, \mathbb{Q}) \subset GL(2, \mathbb{C})$$

is not a matrix group.

Problem Sheet 4

The Exponential Map and Lie Algebra Structure

Exercise 4.1. For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $k \in \mathbb{N}^*$ denote by

$$\mathfrak{n}(k, \mathbb{K}) := \{(a_{ij}) \in M(k \times k, \mathbb{K}) : a_{ij} = 0 \text{ for } j \leqslant i\}$$

the Lie algebra of strictly upper triangular matrices and by

$$UP(k, \mathbb{K}) := \{\mathbb{1} + A \in GL(k, \mathbb{K}) : A \in \mathfrak{n}(k, \mathbb{K})\}$$

the group of unipotent matrices. Show:

(1) The Lie algebra satisfies

$$\mathfrak{up}(k, \mathbb{K}) = \mathfrak{n}(k, \mathbb{K})$$

(2) The exponential map

$$\exp : \mathfrak{up}(k, \mathbb{K}) \to UP(k, \mathbb{K})$$

is surjective and injective.

Exercise 4.2. For the Lie algebra $L := \mathfrak{gl}(n, \mathbb{C})$ consider the adjoint representation

$$\begin{aligned} \mathrm{ad} : L &\to \mathrm{End} L, \\ X &\mapsto \mathrm{ad}_X, \end{aligned}$$

with

$$\begin{aligned} \mathrm{ad}_X : L &\to L, \\ (\mathrm{ad}_X)(Y) &:= [X, Y]. \end{aligned}$$

For $v \in \mathbb{N}$ define the v-th iteration

$$\begin{aligned} (\mathrm{ad}_X)^v &: L \to L, \\ (\mathrm{ad}_X)^v &:= [X, \ldots [X, [X, Y]] \ldots] \end{aligned}$$

with v-times the argument X.

(1) Show by induction

$$(\mathrm{ad}_X)^N(Y) = \sum_{v=0}^{N} \binom{N}{v} X^v \cdot Y \cdot (-X)^{N-v}$$

(2) Define

$$\begin{aligned} \mathrm{e}^{\mathrm{ad}_X} &: L \to L, \\ \mathrm{e}^{\mathrm{ad}_X}(Y) &:= \sum_{N=0}^{\infty} \frac{(\mathrm{ad}_X)^N(Y)}{N!}. \end{aligned}$$

Show

$$(\mathrm{e}^{\mathrm{ad}_X})(Y) = \mathrm{e}^X \cdot Y \cdot \mathrm{e}^{-X}.$$

Exercise 4.3. Consider a pair of two matrices $X, Y \in M(n \times n, \mathbb{C})$ each of which commutes with the commutator, i.e.

$$[X, [X, Y]] = [Y, [X, Y]] = 0.$$

(1) Show the equivalence

$$\exp tX \cdot \exp tY = \exp\left(tX + tY + \frac{t^2}{2} \cdot [X,Y]\right)$$
$$\Updownarrow$$
$$\exp tX \cdot \exp tY \cdot \exp\left(-\frac{t^2}{2} \cdot [X,Y]\right) = \exp(t(X+Y)).$$

(2) Show that the two differentiable functions of the real parameter t

$$\mathbb{R} \to GL(n, \mathbb{C})$$

defined respectively as

$$\exp(t(X+Y)) \text{ and } \exp tX \cdot \exp tY \cdot \exp\left(-\frac{t^2}{2} \cdot [X,Y]\right)$$

satisfy the same linear ordinary differential equation with respect to t and the same initial condition for $t = 0$.

[Hint: You may apply the product rule and combine the three resulting summands by using the functional equation of exp in the commutative case. Then in the first summand the term $X \cdot \exp tY$ can be transformed by the formula from Exercise 4.2.]

(3) Prove the adapted functional equation

$$\exp X \cdot \exp Y = \exp\left(X + Y + \frac{1}{2} \cdot [X,Y]\right).$$

Exercise 4.4. Assume $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and denote by $UP(3,\mathbb{K}) \subset GL(3,\mathbb{K})$ the subgroup of unipotent matrices.

(1) Show: The exponential map

$$\exp : \mathfrak{n}(3,\mathbb{K}) \to UP(3,\mathbb{K})$$

satisfies

$$\exp(X) \cdot \exp(Y) = \exp\left(X + Y + \frac{1}{2} \cdot [X,Y]\right).$$

(2) Define a group structure on $\mathfrak{n}(3,\mathbb{K})$ such that

$$\exp : \mathfrak{n}(3,\mathbb{K}) \to UP(3,\mathbb{K})$$

becomes an isomorphism of groups.

[Hint: You may apply the results of Exercise 4.1 and 4.3.]

Problem Sheet 5

Representations of $\mathfrak{su}(2)$ and $\mathfrak{sl}(2, \mathbb{C})$

Exercise 5.1. For a matrix group G with surjective exponential map

$$\exp : \mathfrak{g} \to G.$$

Show: Each $g \in G$ has for each $n \in \mathbb{N}^*$ a n-th root $\sqrt[n]{g} \in G$, i.e. there exists

$$h \in G \text{ with } h^n = g.$$

Exercise 5.2. For $j = 1, 2, 3$ compute explicitly the value of the 1-parameter subgroup of $SU(2)$ with infinitesimal generator $\mathrm{i} \cdot \sigma_j \in \mathfrak{su}(2)$ with the Pauli matrix σ_j.

Exercise 5.3. Assume the following results:

- For each representation of $\mathfrak{su}(2)$ on a finite dimensional complex vector space V

  $$\lambda : \mathfrak{su}(2) \to \mathfrak{gl}(V)$$

  exists a unique morphism of matrix groups

  $$\Lambda : SU(2) \to GL(V)$$

  such that the following diagram commutes

  $$\begin{tikzcd} SU(2) \arrow[r, dashed, "\Lambda"] & GL(V) \\ \mathfrak{su}(2) \arrow[u, "\exp"] \arrow[r, "\lambda"] & \mathfrak{gl}(V) \arrow[u, "\exp"'] \end{tikzcd}$$

- For each $n \in \mathbb{N}$ exists a representation

  $$\rho_n : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{gl}(V_n)$$

  with an $(n+1)$-dimensional complex vector space

  $$V_n = \mathrm{span}_{\mathbb{C}} \langle e_0, \ldots, e_n \rangle$$

  and the $\mathfrak{sl}(2, \mathbb{C})$-action: For $j = 0, \ldots, n$

  $$h.e_j = (n - 2j) \cdot e_j, \; x.e_j = (n - j + 1) \cdot e_{j-1}, \; y.e_j = (j+1) \cdot e_{j+1};$$
  $$e_{n+1} := e_{-1} := 0$$

  for the elements

  $$h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad x := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{C})$$

Define the restriction to $\mathfrak{su}(2)$ as

$$\lambda_n := \rho_n|_{\mathfrak{su}(2)} : \mathfrak{su}(2) \to \mathfrak{gl}(V_n), n \in \mathbb{N}.$$

Show the equivalence of the following two properties:

- The parameter $n \in \mathbb{N}$ is even.
- For the induced morphism of matrix groups

$$\Lambda_n : SU(2) \to GL(V_n)$$

with tangent map λ_n exists a morphism of matrix groups

$$\overline{\Lambda}_n : SO(3,\mathbb{R}) \to GL(V_n)$$

such that the following diagram——with Φ the universal covering——commutes

$$\begin{tikzcd} SU(2) \arrow[r, "\Lambda_n"] \arrow[d, "\Phi"'] & GL(V_n) \\ SO(3,\mathbb{R}) \arrow[ur, dashed, "\overline{\Lambda}_n"'] & \end{tikzcd}$$

Note: The group morphisms Λ_n with odd $n \in \mathbb{N}$ are named the spinor representations of $SU(2)$.

Exercise 5.4. Consider a Lie algebra L and an ideal $I \subset L$. Assume: The Lie algebra L/I is nilpotent and for all $x \in L$ the restricted endomorphism

$$(\mathrm{ad}\ x)|_I : I \to I$$

is nilpotent. Show: The Lie algebra L is nilpotent.

Problem Sheet 6

Nilpotent Lie Algebras

Exercise 6.1. Consider the following diagram with two short exact sequences of morphisms of Lie algebras, and assume the existence of a morphism

$$f : L_1 \to L_3$$

which makes the diagram commutative:

$$\begin{tikzcd}
 & & L_1 \arrow[dd, dashed, "f"] \arrow[dr, "\pi_1"] & & \\
0 \arrow[r] & L_0 \arrow[ur, "j_1"] \arrow[dr, "j_2"'] & & L_2 \arrow[r] & 0 \\
 & & L_3 \arrow[ur, "\pi_2"'] & &
\end{tikzcd}$$

Show that f is an isomorphism of Lie algebras.

Exercise 6.2. Consider a Lie algebra L. Show:

(1) For two nilpotent ideals $I, J \subset L$ also the sum $I + Y \subset L$ is a nilpotent ideal.

(2) There exists a unique maximal nilpotent ideal in L (named the nilradical of L).

Exercise 6.3. Consider a nilpotent $\mathbb{K}$-Lie algebra $L \neq \{0\}$. Show:

(1) There exists a $\mathbb{K}$-vector space decomposition

$$L = I \oplus \mathbb{K} \cdot x_0$$

with an ideal $I \subset L$ and a non-zero element $x_0 \in L$.

(2) The centralizer of I satisfies

$$C_L(I) \neq \{0\},$$

and there exists a maximal exponent $n \in \mathbb{N}$ with

$$C_L(I) \subset C^n L.$$

(3) There exists an outer derivation of L, i.e. a derivation

$$D : L \to L$$

which does not have the form

$$D = \text{ad } u \text{ with } u \in L.$$

[Hint: You may use $C_L(I) \setminus C^{n+1} L \neq \varnothing$.]

Exercise 6.4. Determine explicitly an outer derivation of the Lie algebra in $\mathfrak{n}(3, \mathbb{K})$.

Problem Sheet 7

The Heisenberg Algebra and Bilinear Forms

Exercise 7.1. Determine the center $Z(\mathfrak{h}(n))$ of the Heisenberg algebra.

Exercise 7.2. For $n \in \mathbb{N}$ consider the vector space of square matrices

$$M := M(n \times n, \mathbb{K}),$$

and the symmetric bilinear trace form

$$\beta : M \times M \to \mathbb{K},$$
$$\beta(A, B) := \operatorname{tr}(A \cdot B)$$

For each subspace $V \subset M$ denote by

$$V^{\perp} := \{A \in M : \beta(A, v) = 0 \text{ for all } v \in V\}$$

the orthogonal space of V. Show:

(1) The form β is non-degenerate, i.e. $M^{\perp} = \{0\}$.

(2) The canonical map to the dual space

$$j_{\beta} : M \to M^{*},$$
$$A \mapsto \beta(A, -),$$

is an isomorphism of $\mathbb{K}$-vector spaces.

(3) For each vector subspace $V \subset M$ holds

$$j_{\beta}(V^{\perp}) = V^{0} := \{\lambda \in M^{*} : \lambda|_{V} = 0\}$$

and j_{β} induces an isomorphism

$$M/V^{\perp} \xrightarrow{\simeq} V^{*}.$$

Exercise 7.3. For $n \in \mathbb{N}$ consider the group

$$AF(n, \mathbb{K}) := \{\mathbb{K}^{n} \to \mathbb{K}^{n}, v \mapsto A \cdot v + b : A \in GL(n, \mathbb{K}), b \in \mathbb{K}^{n}\}$$

of affine automorphisms of $\mathbb{K}^{n}$.

(1) Show: The group $AF(n, \mathbb{K})$ is isomorphic to a matrix group $G \subset GL(n+1, \mathbb{K})$. In the following identity

$$AF(n, \mathbb{K}) \text{ and } G.$$

(2) Compute the Lie algebra $\mathfrak{af}(n, \mathbb{K})$ of Lie group $AF(n, \mathbb{K})$.

(3) Show that $\mathfrak{af}(n, \mathbb{K})$ is a semidirect product

$$I \rtimes_{\theta} M$$

with two $\mathbb{K}$-Lie algebras I and M, and a suitable morphism of Lie algebras

$$\theta : M \to \operatorname{Der}(I).$$

Exercise 7.4. Consider the Lie algebra $\mathfrak{sl}(2,\mathbb{K})$ and its standard basis $(e_i)_{i=1,2,3}$ with

$$e_1 := h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ e_2 := x := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ e_3 := y := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{K}).$$

(1) With respect to the standard basis compute the matrices from $M(2\times 2,\mathbb{K})$ of the endomorphisms of $\mathfrak{sl}(2,\mathbb{K})$

$$\mathrm{ad}\ h,\ \mathrm{ad}\ x,\ \mathrm{ad}\ y.$$

(2) Determine the Killing form of $\mathfrak{sl}(2,\mathbb{K})$ with respect to the standard basis, i.e. determine the matrix

$$Q = (\kappa(e_i,e_j))_{1\leqslant i,j\leqslant 3}.$$

(3) Determine the rank and the eigenvalues of Q.

Problem Sheet 8

Review of Key Concepts and Definitions

Exercise 8.1. Which classes of Lie algebras do you know? Give the definition of each class.

Exercise 8.2. Is any nilpotent Lie algebra also solvable?

Exercise 8.3. What is the content of the Cartan criterion for solvability?

Exercise 8.4. What does Lie's theorem state, why does one need the complex numbers as base field?

Exercise 8.5. How is the Killing form defined? Give some applications of the Killing form.

Exercise 8.6. Set

$$A = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}, \ \lambda \in \mathbb{K}.$$

Determine the minimal polynomial $p_{\min}(T)$ of A and its characteristic polynomial $p_{\text{char}}(T)$. How do they relate?

Exercise 8.7. Give the definition of the trace form of a representation?

Exercise 8.8. Set

$$B = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}.$$

Why are the matrices

$$B_s = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \text{ and } B_n = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

not the summands of the Jordan decomposition of B?

Exercise 8.9. What is the content of the Cartan criterion for semisimpleness?

Exercise 8.10. State the Jacobi identity.

Exercise 8.11. State the definition of a representation of a Lie algebra.

Exercise 8.12. For which class of Lie algebras is the adjoint representation faithful?

Exercise 8.13. State the definition and some properties of the exponential map of matrices.

Exercise 8.14. Give an example of an infinite matrix series. For which matrices does the series converge?

Exercise 8.15. State the difference between the matrix product and the Lie bracket of Lie algebras.

Exercise 8.16. What are derivations, how do they relate to the adjoint representation?

Exercise 8.17. State the definition and name some properties of the Killing form.

Exercise 8.18. How do $p_{\min}(T)$ and $p_{\text{char}}(T)$ relate for general square matrices?

Exercise 8.19. What about surjectivity of the exponential map?

Exercise 8.20. State the definition and some properties of the Heisenberg Lie algebra.

Exercise 8.21. State the main theorem about nilpotent Lie algebras.

Exercise 8.22. Define the semidirect product of two Lie algebras. How does it relate to the direct product?

Exercise 8.23. How does a semisimple Lie algebra split?

Exercise 8.24. Give the definition of the orthogonal space of an ideal in a semisimple Lie algebra, and state its property.

Exercise 8.25. State the definition of the Lie algebra of a matrix group.

Exercise 8.26. How is the adjoint representation defined?

Exercise 8.27. State the main theorem about solvable Lie algebras.

Exercise 8.28. Name some of the classical matrix groups and derive their Lie algebras.

Exercise 8.29. State the definition of a 1-parameter group.

Exercise 8.30. State the definition of a connected topological space.

Exercise 8.31. Describe the universal covering projection of $SO(3, \mathbb{R})$.

Exercise 8.32. How does the dynamic Lie algebra of quantum mechanics relate to the Heisenberg algebra?

Exercise 8.33. State the general form of nilpotent matrix Lie algebras?

Exercise 8.34. State the definition of the fundamental group of a connected topological space.

Exercise 8.35. State Weyl's theorem on complete reducibility.

Exercise 8.36. Describe the universal covering projection of the identity component of the Lorentz group.

Exercise 8.37. How does respectively nilotency and solvability behave in short exact sequences of Lie algebra morphisms?

Exercise 8.38. Characterize semisimpleness of a Lie algebra by its radical.

Exercise 8.39. When does a short exact sequence of Lie algebra morphisms split? What does splitting imply?

Exercise 8.40. Name some applications of the Jordan decomposition in Lie algebra theory.

Exercise 8.41. State some types of induced representations. Prove that they are representations.

Exercise 8.42. Give some examples from classical matrix groups which are simply connected and others which are not simply connected.

Problem Sheet 9

Semisimple Lie Algebras and Their Structure

Exercise 9.1. For the Lie algebra

$$L := \mathfrak{sl}(n, \mathbb{C}),\ n \in \mathbb{N},$$

Show: The subalgebra of diagonal matrices

$$\mathfrak{d}(n, \mathbb{C}) \cap L$$

is a maximal toral subalgebra of L.

Exercise 9.2. Consider a simple complex Lie algebra L and two bilinear symmetric forms

$$\beta, \gamma : L \times L \to \mathbb{C}$$

which are non-degenerate and satisfy for $x,\ y,\ z \in L$ the "associativity"

$$\beta([x,y],z) = \beta(x,[y,z]), \gamma([x,y],z) = \gamma(x,[y,z]).$$

Show: There exists a scalar $\mu \in \mathbb{C}^*$ satisfying

$$\beta = \mu \cdot \gamma.$$

Exercise 9.3. For

$$L := \mathfrak{sl}(2, \mathbb{C})$$

consider the Killing form κ and the trace form

$$\mathrm{tr} : L \times L \to \mathbb{C},$$
$$\mathrm{tr}(x, y) := \mathrm{tr}(x \circ y).$$

Determine $\mu \in \mathbb{C}^*$ with

$$\kappa = \mu \cdot \mathrm{tr}$$

Exercise 9.4.

(1) For an Abelian Lie algebra I, show: Each endomorphism of the vector space I is a derivation of the Lie algebra I, i.e.

$$\mathfrak{gl}(I) = \mathrm{Der}(I).$$

(2) Consider a Lie algebra S and an Abelian Lie algebra I. Due to part (1) each representation

$$\rho : S \to \mathfrak{gl}(I)$$

satisfies $\rho(S) \subset \mathrm{Der}(I)$. Therefore the semidirect product

$$L := I \rtimes_\rho S$$

is a well-defined Lie algebra, fitting into the exact sequence of Lie algebras

$$0 \to I \xrightarrow{j} L \xrightarrow{\pi} S \to 0.$$

Denote by

$$s : S \to L$$

a section against π. Assume S semisimple, and the representation $\rho : S \to \mathfrak{gl}(I)$ non-zero and irreducible. Show:

(a) Derived algebra: $L = [L, L]$.

(b) Center: $Z(L) = \{0\}$.

(c) No factorizing as direct product: There does not exist a pair (L_1, L_2) of Lie algebras with L_1 semisimple and L_2 solvable, such that

$$L \simeq L_1 \times L_2.$$

In particular, L is not semisimple.

[Hint: (i) Consider $I \subset L$, $S \subset L$ and verify $\rho(S)(I) = I$. Conclude $[I, S]_L = I$. Show $[S, S]_L = S$. (ii) From $(i, s) \in Z(L)$ conclude $s = 0$.]

Problem Sheet 10

Representation Theory of $\mathfrak{sl}(2, \mathbb{C})$

We set $L := \mathfrak{sl}(2, \mathbb{C})$ for all problems on the present problem sheet.

Exercise 10.1. Consider two L-modules U and W. Show: If $u \in U$ is a weight vector of weight λ_u and $w \in W$ a weight vector of weight λ_w, then the tensor product

$$u \otimes w \in U \otimes_{\mathbb{C}} W$$

is a weight vector of weight $\lambda_u + \lambda_w$.

Exercise 10.2. Consider the injection of Lie algebras

$$\begin{aligned} j : L &\hookrightarrow \mathfrak{sl}(3, \mathbb{C}), \\ A &\mapsto j(A) := \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

considered as a block matrix.

(1) Show: With respect to the representation

$$\begin{aligned} \rho : L &\to \mathfrak{gl}(\mathfrak{sl}(3, \mathbb{C})), \\ z &\mapsto \text{ad } j(z), \end{aligned}$$

the L-module $\mathfrak{sl}(3, \mathbb{C})$ is reducible.

(2) Why is the L-module $\mathfrak{sl}(3, \mathbb{C})$ from part (1) completely reducible? Determine the isomorphism classes of the irreducible L-modules from the splitting of $\mathfrak{sl}(3, \mathbb{C})$.

Exercise 10.3. Denote by $V(\lambda)$ the irreducible L-module with highest weight λ.

Determine the irreducible components of the L-module

$$V(4) \otimes_{\mathbb{C}} V(7).$$

Exercise 10.4. For arbitrary p, $q \in \mathbb{Z}_+$ determine the weights of the L-module

$$V(p) \otimes_{\mathbb{C}} V(q)$$

and the dimension of their weight spaces.

Problem Sheet 11

Root Systems of $\mathfrak{so}(4,\mathbb{C})$ and $\mathfrak{sl}(3,\mathbb{C})$

Exercise 11.1.

(1) Show the isomorphy of complex Lie algebras

$$\mathfrak{so}(4,\mathbb{C}) \simeq \mathfrak{so}(3,\mathbb{C}) \oplus \mathfrak{so}(3,\mathbb{C}).$$

(2) Determine the root sets of $\mathfrak{so}(3,\mathbb{C})$ and of $\mathfrak{so}(4,\mathbb{C})$, the root space decomposition of $\mathfrak{so}(4,\mathbb{C})$, and explicit generators of each root space of $\mathfrak{so}(4,\mathbb{C})$.

(3) Determine the rank and a base of the root systems of $\mathfrak{so}(3,\mathbb{C})$ and $\mathfrak{so}(4,\mathbb{C})$.

[Hint: (i) Define a suitable injective map $\mathfrak{so}(3,\mathbb{C}) \oplus \mathfrak{so}(3,\mathbb{C}) \to \mathfrak{so}(4,\mathbb{C})$. (ii) Use suitable generators of $\mathfrak{so}(3,\mathbb{C})$.]

For the following Exercise 11.2 and 11.3, set $L := \mathfrak{sl}(3,\mathbb{C})$.

Exercise 11.2.

(1) Choose a maximal total subalgebra $T \subset L$ and determine explicitly a vector space basis $(h_j)_{j\in I}$ of T.

(2) Determine the root set Φ of L with respect to T and a base Δ of the root system $R = (\mathbb{R}^2, \Phi)$.

(3) Compute the root space decomposition of L: For each positive root $\alpha \in \Phi^+$ determine root vectors

$$x_\alpha \in L^\alpha,\ y_\alpha \in L^{-\alpha}$$

such that the subalgebra of L

$$L_\alpha := \mathrm{span}_{\mathbb{C}}\langle x_\alpha, y_\alpha, h_\alpha := [x_\alpha, y_\alpha]\rangle$$

is isomorphic to

$$\mathfrak{sl}(2,\mathbb{C}).$$

Exercise 11.3. Denote by Φ the root set of L with respect to a maximal toral subalgebra $T \subset L$, and by

$$V := \mathrm{span}_{\mathbb{R}}\Phi$$

the real vector space spanned by the roots $\alpha \in \Phi$.

(1) Determine the rank of the root system $R := (V, \Phi)$ of L.

(2) Determine the Cartan matrix of R.

Exercise 11.4. Compute the Weyl group of the root system of $\mathfrak{so}(4,\mathbb{C})$ and of the root system of $\mathfrak{sl}(3,\mathbb{C})$.

Problem Sheet 12

Review in Representation Theory

Exercise 12.1. Name a maximal toral subalgebra of $\mathfrak{sl}(2,\mathbb{C})$ and more general of $\mathfrak{sl}(n,\mathbb{C})$.

Exercise 12.2. What is a Cartan integer?

Exercise 12.3. Describe the irreducible finite-dimensional $\mathfrak{sl}(2,\mathbb{C})$-modules.

Exercise 12.4. Describe the structure of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$.

Exercise 12.5. Which role plays the Lie algebra $\mathfrak{so}(3,\mathbb{C})$ in physics?

Exercise 12.6. How to obtain all complex representations of the matrix group $SO(3,\mathbb{R})$?

Exercise 12.7. What is a base of a root system? Why is the concept important?

Exercise 12.8. What is a primitive element, and why is the concept important?

Exercise 12.9. In which respect differ the Coxeter graph and the Dynkin diagram of a root system?

Exercise 12.10. How do the Lie algebras $\mathfrak{su}(n)$ and $\mathfrak{sl}(n,\mathbb{C})$ relate to each other?

Exercise 12.11. Is the base of a root system uniquely determined?

Exercise 12.12. What are ladder operators?

Exercise 12.13. Define the concept of a root system. Why are root systems important?

Exercise 12.14. Define the Lie algebra of the angular momentum and its commutator relations.

Exercise 12.15. Which conditions on two bases of a root system ensure that they define isomorphic root systems?

Exercise 12.16. How do the Lie algebras $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{so}(3,\mathbb{C})$ relate to each other?

Exercise 12.17. How to obtain all representations of the matrix group $SU(n)$?

Exercise 12.18. Write down the Cartan matrices of bases of $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{so}(4,\mathbb{C})$. Explain their form.

Exercise 12.19. Which concept is the Weyl group of a root system, and why is the concept important?

Exercise 12.20. Determine the weight spaces of an irreducible finite-dimensional $\mathfrak{sl}(2,\mathbb{C})$-module and their dimensions.

Exercise 12.21. Which Cartan integers are possible for the root system of a semisimple complex Lie algebra?