Differentiable Manifolds Problem Sheets

Prof. D. Kotschick
Dr. J. Stelzig

Contents

Problem Sheet 1

Topology

1.1 Compactness

Exercise 1.1. Recall that a topological space X is called compact if every open covering of X has a finite subcovering. A subset of a topological space is called compact if it is compact in the subspace topology. Let X, Y be topological spaces. Prove

1. If X is compact and $A \subset X$ is closed, then A is compact.
2. If $f : X \longrightarrow Y$ is continuous and X compact, then $f(X) \subset Y$ is compact.
3. If X is Hausdorff and $A \subset X$ is compact, then A is closed.

1.2 Manifold has a countable basis with compact closures

Exercise 1.2. Let M be a topological manifold. Show that there exists a countable basis of the topology consisting of open sets with compact closures (note that the definition of a manifold asks for a countable basis without other conditions).

1.3 Metric Space is Hausdorff

Exercise 1.3. Show that any metric space is Hausdorff.

1.4 A Homeomorphism which is not A Diffeomorphism

Exercise 1.4. Give an example of a homeomorphism which is not a diffeomorphism.

1.5 Stereographic Projection

Exercise 1.5. The n-sphere S^n is defined as the subspace

$$S^n := \left\{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_n^2 = 1\right\}$$

Consider the north-pole $N = (0, 0, \dots, 0, 1) \in S^n$. The stereographic projection from N is the map $\phi_N : S^n \backslash \{N\} \to \mathbb{R}^n$ which takes a point $p \in S^n \backslash \{N\}$ to the intersection of the line through N and p with the hyperplane $\mathbb{R}^n = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$. There is a similar map ϕ_S for the south-pole $S = (0, 0, \dots, 0, -1)$.

Write down coordinate expressions for ϕ_N and ϕ_S and use these to show S^n is a topological manifold.

Problem Sheet 2

Differentiable Manifold

2.1 Mapping Tori

Exercise 2.1. Let F be a differentiable manifold and $\phi : F \to F$ a diffeomorphism. Consider the topological space $F \times \mathbb{R}$ with the equivalence relation $\sim$ given by

$$(x,t) \sim (y,s) \iff (\phi^n(x), t+n) = (y,s) \text{ for some } n \in \mathbb{Z}$$

The quotient space $M := (F \times \mathbb{R})/\sim$ is called the mapping torus of ϕ. Important examples of mapping tori are the 2-torus T^2 $\big(F = S^1, \phi(x) = x\big)$, the Klein bottle K^2 $\big(F = S^1, \phi(x) = -x\big)$ and the twisted 2-sphere bundle over the circle, $S^2 \tilde{\times} S^1$ $\big(F = S^2, \phi(x) = -x\big)$.

Show that $\sim$ really is an equivalence relation. The quotient by ϕ is clearly Hausdorff and has a countable basis of the topology. Show that M is a differentiable manifold by explicitly defining a differentiable structure on M, induced from that of F and $\mathbb{R}$.

2.2 Projective Space

Exercise 2.2. Let $\mathbb{K}$ denote $\mathbb{R}$ or $\mathbb{C}$. Define an equivalence relation on $\mathbb{K}^{n+1}\backslash\{0\}$ as follows:

$$x \sim y \iff x = \lambda y \text{ for some } \lambda \in \mathbb{K}\backslash\{0\}.$$

The (real or complex) projective n-space is defined to be the quotient space

$$\mathbb{K}^n := \big(\mathbb{K}^{n+1}\backslash\{0\}\big) / \sim .$$

The equivalence class of a point $(x_0, \ldots, x_n)$ is denoted by $[x_0 : \cdots : x_n]$.

(1) Show that $\mathbb{KP}^n$ is a compact differentiable manifold of dimension $n(\mathbb{K} = \mathbb{R})$, resp. $2n(\mathbb{K} = \mathbb{C})$. You can use the sets $U_i := \{[x_0 : \cdots : x_n] \mid x_i \neq 0\}$ for $i = 0, \ldots, n$ as domains of charts.

(2) Consider the projection map

$$\begin{aligned} \pi : S^n &\longrightarrow \mathbb{K}^n \\ (x_0, \ldots, x_n) &\longmapsto [x_0 : \cdots : x_n] . \end{aligned}$$

Show that π is a smooth map whose derivative at every point is surjective. What is the preimage of a point in $\mathbb{KP}^n$?

2.3 Diffeomorphism between the Complex Projective Line and S^2

Exercise 2.3. Consider the case $\mathbb{K} = \mathbb{C}$, $n = 1$ from the Exercise 2.2. This is a differentiable manifold $\mathbb{CP}^1 = \{[z_0 : z_1] \mid z_i \in \mathbb{C}\}$. Using the charts from the Exercise 2.2 and those for S^2 from Exercise 1.5, construct a diffeomorphism from $\mathbb{CP}^1$ to S^2.

2.4 The General Linear Group as a Lie Group

Exercise 2.4. Let $\mathbb{K}$ again denote $\mathbb{R}$ or $\mathbb{C}$. Write $GL_n(\mathbb{K}) := \{A \in \mathbb{K}^{n\times n} \mid \det(A) \neq 0\}$. Show that $GL_n(\mathbb{K})$ is a differentiable manifold and that multiplication an formation of inverses define smooth maps $GL_n(\mathbb{K}) \times GL_n(\mathbb{K}) \to GL_n(\mathbb{K})$, resp. $GL_n(\mathbb{K}) \to GL_n(\mathbb{K})$.

Problem Sheet 3

Tangent Spaces and Tangent Bundle

3.1 Constant Rank Theorem and Local Coordinate Representation

Exercise 3.1. Let $f : M \to N$ be a smooth map of smooth manifolds and assume there is a neighborhood of $p \in M$ on which Df is of constant rank k. Show that there exist charts (U, φ) around p a and (V, ψ) around $q := f(p)$ such that $\psi \circ f \circ \varphi$ has the form

$$(x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_k, 0, \ldots, 0)\,.$$

3.2 Embedding of Submanifolds via Inclusion Maps

Exercise 3.2. Show that for any submanifold $Z \subseteq M$ of a smooth manifold, the inclusion map $i : Z \hookrightarrow M$ is an embedding.

3.3 Rank Analysis of the Height Function on S^n

Exercise 3.3. Cousider $S^n := \left\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum x_i^2 = 1\right\}$ and the "height" function

$$\begin{aligned} h : S^n &\longrightarrow \mathbb{R} \\ (x_0, \ldots, x_n) &\longmapsto x_n. \end{aligned}$$

Compute Dh in terms of the stereographic coordinates from Exercise 1.5 and determine its rank at every point.

3.4 Regularity Criteria for Projective Algebraic Sets

Exercise 3.4. Consider a collection of homogeneous polynomials $p_1, \ldots, p_k \in \mathbb{R}\left[T_0, \ldots, T_n\right]$. Convince yourself that the projective vanishing set

$$V\left(p_1, \ldots, p_k\right) := \left\{[z_0 : \cdots : z_n] \in \mathbb{RP}^n \mid p_1\left(z_0, \ldots, z_n\right) = \cdots = p_k\left(z_0, \ldots, z_n\right) = 0\right\} \subseteq \mathbb{RP}^n$$

is well defined and find a necessary criterion in terms of the partial derivatives of the p_j for this set to be a submanifold of $\mathbb{RP}^n$.

Problem Sheet 4

Vector Bundles

4.1 Local Constancy of Fiber Rank in Vector Bundles

Exercise 4.1. A smooth map $f : E \to F$ between two vector bundles $\pi_E : E \to M, \pi_F : F \to M$ over a smooth manifold M is called a bundle map if $\pi_E = \pi_F \circ f$ and for each $x \in M$ the restriction to the fibre $f_x : E_x \to F_x$ is linear.

(1) Show that there is an open subset of $U \subseteq M$ such that the fibre-wise rank of $f\Big|_U$ is constant.

(2) Give an example where the rank is not constant on all of M.

4.2 Sard's Theorem for Compact Manifolds of Lower Dimension

Exercise 4.2. Let $f : M \to N$ be a smooth map of smooth manifolds such that $\dim M < \dim N$ and M is compact. Show Sard's theorem in this case, i.e. that the complement of the image of f is open and dense, using the following intermediate steps:

(1) Show that the image of f is closed.

(2) Show that there is an open set $U \subseteq M$ such that the rank of Df is constant on U.

Then use the local form for maps of constant rank from last sheet to conclude.

4.3 Matrix Lie Groups as Embedded Submanifolds

Exercise 4.3. Show that $SO(n)$, $O(n)$, $SL_n(\mathbb{R})$ are submanifolds of $GL_n(\mathbb{R})$ and compute their dimensions.

4.4 Nonvanishing Vector Fields on Odd-Dimensional Spheres

Exercise 4.4. Using the description $TS^n = \left\{(x, v) \in S^n \times \mathbb{R}^{n+1} \mid \langle x, v\rangle = 0\right\}$ for the tangent bundle of the sphere, show that there always exist a nowhere vanishing section of TS^n when n is odd. (Hint: $\mathbb{R}^{2m} \cong \mathbb{C}^m$).

Problem Sheet 5

Vector Bundles and Dynamical Systems

5.1 Kernel, Image, and Quotient Subbundles

Exercise 5.1. A smooth map $f : E \to F$ between two vector bundles $\pi_E : E \to M$, $\pi_F : F \to M$ over a smooth manifold M is called a bundle homomorphism if $\pi_E = \pi_F \circ f$ and for each $x \in M$ the restriction to the fibre $f_x : E_x \to F_x$ is linear.

(1) Prove that if the rank of f_x is constant, $\ker f := \{e \in E \mid f(e) = 0\}$ and $\operatorname{im} f := \{f(e) \mid e \in E\}$ are subbundles of E, resp. F.

(2) Let $E \subseteq F$ be a subbundle. Show that one obtains a well-defined smooth vector bundle F/E which over a point $x \in M$ has the fibre F_x/E_x, i.e. the quotient of the fibres over x.

5.2 Tangent Bundle Structure and Pullback Trivialization

Exercise 5.2.

(1) Let $\pi : E \to X$ be a vector bundle. Show that there is a bundle isomorphism $TE = \pi^*(TX \oplus E)$.

(2) Let $\pi : M \to S^1$ be the Möbius strip. Let $f : S^1 \to S^1$ be defined by $z \mapsto z^2$, where $S^1 = \{z \in \mathbb{C} \mid \|z\| = 1\}$. Show that f^*M is the trivial bundle.

5.3 Diverse Flow Regimes on the 2-Torus

Exercise 5.3. Construct distinct flows on the 2-torus $M = S^1 \times S^1$ with the following properties:

(1) All of the flow lines are closed

(2) Some flowlines are closed and others are not

(3) None of the flowlines are closed

You may use the description of the torus as $M \cong \mathbb{R}^2/\mathbb{Z}^2$.

5.4 Integrating Linear Vector Fields on the Plane

Exercise 5.4. Find a flow on $\mathbb{R}^2$ which has the following velocity vector field:

(1) $V = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}$

(2) $V = -y\dfrac{\partial}{\partial x} + x\dfrac{\partial}{\partial y}$

Problem Sheet 6

Lie Brackets, Lie Algebras, and Lie Groups

6.1 Jacobi Identity and the Lie Bracket Structure

Exercise 6.1. Let M be a smooth manifold. Show that the Lie bracket satisfies the Jacobi identity, i.e. for any three vector fields X, Y, Z on M, one has an equality

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0.$$

6.2 Coordinate Vector Fields and Lie Bracket Computations

Exercise 6.2. Let (U, φ) be a chart of an n-dimensional smooth manifold M. Let e_i denote the coordinate vector fields, which correspond to the derivations $\frac{\partial}{\partial x_i}$, where $x_i : U \to \mathbb{R}$ are the coordinate functions of φ, i.e. $\varphi(p) = (x_1(p), \dots, x_n(p))$.

(1) Show that $e_1, \dots, e_n$ form a basis at the tangent space T_pU for every $p \in U$ and that $[e_i, e_j] = 0$.

(2) Let $f, g : M \to \mathbb{R}$ be smooth functions. Show that

$$[fX, gY] = f\,(L_X g)\,Y - g\,(L_Y f)\,X + fg[X,Y]$$

for all vector fields X, Y on M.

(3) Calculate as an example the Lie bracket of the vector fields $X = x^2 y e_1 + e_2$ and $Y = x e_1 - y^2 e_2$ on $\mathbb{R}^2$ with coordinates (x, y), i.e. $e_1 = \frac{\partial}{\partial x}$ and $e_2 = \frac{\partial}{\partial y}$.

6.3 Diffeomorphisms, Push-Forwards, and Invariant Vector Fields

Exercise 6.3. Let M be a smooth manifold. For $\phi : M \to N$ a diffeomorphism and X a vector field on M, we define the push-forward of X via $(\phi_* X)(\phi(p)) := (D_p\phi)(X(p))$.

(1) For X, Y vector fields on M, show that

$$\phi_*[X,Y] = [\phi_s X, \phi_* Y]$$

(2) Deduce that for M a Lie group and X, Y left-invariant vector fields, also $[X, Y]$ is left-invariant, i.e. the left-invariant vector fields form a Lie algebra.

6.4 Lie Algebras of Classical Matrix Groups

Exercise 6.4. Let G be a Lie group. Identify the Lie algebra of left-invariant vector fields with the tangent space at the neutral element $e \in G$. Describe the Lie algebras of $GL_n(\mathbb{R})$, $SL_n(\mathbb{R})$, $O(n)$ and $SO(n)$.

Problem Sheet 7

Vector Fields, Bundles, and Integrability

7.1 Flows, Diffeomorphisms, and Homogeneity of Manifolds

Exercise 7.1.

(1) Consider the balls of radius 1 and 2, $B_1(0) \subseteq B_2(0) \subseteq \mathbb{R}^n$. For any $y \in B_1(0)$, construct a complete vector field X on $B_2(0)$ for which the associated global flow Φ_t satisfies $\Phi_1(0) = y$.

(2) Use the previous item to show that for a connected, differentiable manifold M and any two given points $x, y \in M$, there is a diffeomorphism $\phi : M \to M$ such that $\phi(x) = y$.

7.2 Submersions and Integrable Distributions

Exercise 7.2. Let $p : M \to N$ be a submersion. Show that $\ker Dp \subseteq TM$ is an integrable subbundle.

7.3 Pullback Bundles and Tangent Splittings on Product Manifolds

Exercise 7.3. Let $\pi : E \to M$ be a vector bundle and $f : N \to M$ a smooth map. Let $f^*E = \{(p, v) \in N \times E \mid f(p) = \pi(v)\}$ be the pullback bundle.

(1) Given a cocycle for E, compute a cocycle for f^*E.

(2) Prove that on a product of two smooth manifolds $M = X \times Y$ with projection maps p_X, p_Y to the factors, there is an isomorphism $p_X^* T_X \oplus p_Y^* T_Y \cong TM$, under which the two summands map to integrable subbundles.

7.4 Non-Integrable Distributions and the Frobenius Theorem

Exercise 7.4. Consider a subbundle $V \subseteq T\mathbb{R}^3$ defined as follows: If x, y, z are the coordinates on $\mathbb{R}^3$, the fibre V_p at any point is spanned by the values $X_1(p), X_2(p)$ of the sections $X_1 = \dfrac{\partial}{\partial y}$ and $X_2 = \dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial z}$.

(1) Draw pictures of V_p for various points $p \in \mathbb{R}^3$.

(2) Show that V is not integrable.

Problem Sheet 8

Exterior Algebra, Bilinear Forms, and Line Bundles

8.1 Determinants and Induced Maps on Exterior Powers

Exercise 8.1. Let V be a vector space of finite dimension n and $f : V \to V$ a linear map. Show that the induced map $\lambda(f) : \Lambda^n(V) \to \Lambda^n(V)$ is multiplication by the determinant $\det(f)$.

8.2 Canonical Forms for Skew-Symmetric Bilinear Forms

Exercise 8.2. Let V be a vector space of finite dimension n and $\omega \in \Lambda^2(V^*) \cong (\Lambda^2 V)^*$ be a 2-form, i.e. an skew-symmetric bilinear map $V \times V \to \mathbb{R}$. Given a basis $e_1, \dots e_n$ of V, one can describe ω by a matrix with entries $\omega_{ij} = \omega(e_i, e_j)$.

(1) Let $n = 2$ and $\omega \neq 0$. Show that there exists a basis e_1, e_2 of V such that the matrix of ω with respect to this basis has the form

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(2) For n arbitrary, show that there exists a basis $e_1, \dots e_n$ and a number k with $2k \leqslant n$ such that

$$\omega = \sum_{i=1}^{k} \alpha_{2i-1} \wedge \alpha_{2i}$$

where $\alpha_1, \dots, \alpha_n$ denotes the dual basis of V^*. (For the proof, you can consider the subspaces $W' = \{v \in V \mid \omega(v, w) = 0, \forall w \in W\}$ for subspaces $W \subseteq V$.)

8.3 Decomposability of 2-Forms and the Wedge-Square Criterion

Exercise 8.3. Let V be a vector space of finite dimension n. A 2-form $\omega \in \Lambda^2(V^*)$ is called decomposable if there exist 1-forms $\alpha, \beta \in V^* = \Lambda^1(V^*)$ such that $\omega = \alpha \wedge \beta$. Using the previous exercise, show that ω is decomposable iff $\omega \wedge \omega = 0$. If $n \geqslant 4$, find a 2-form which is not decomposable.

8.4 Tensor Products and Line Bundle Group Structure

Exercise 8.4.

(1) Show that for any two finite-dimensional vector spaces V, W there is a canonical isomorphism $V^* \otimes W \cong \mathrm{Hom}(V, W)$ where the right hand side denotes the linear maps from V to W.

(2) Let M be a manifold. A line bundle $L \to M$ is a rank 1 vector bundle over M. Prove that the set $\mathcal{L}$ of isomorphism classes of line bundles over M is an abelian group with multiplication given by tensor product.

Problem Sheet 9

Differential Forms, Orientations, and Integrability

9.1 Orientations and Symplectic Structures on Vector Spaces

Exercise 9.1. Let V be a vector space of finite dimension $n > 0$. An orientation of V consists of an equivalence class of ordered bases under basis changes by matrices with positive determinant.

(1) Show that V has precisely two orientations and that any orientation of V induces one on $\Lambda^n(V^*)$ and vice versa.

(2) A 2-form $\omega \in \Lambda^2(V^*)$ is called non-degenerate or symplectic if for each non-zero vector $x \in V$ there exists a $y \in V$ such that $\omega(x, y) \neq 0$. Prove that symplectic forms can exist only for n even.

(3) If $n = 2k$ and $\omega \in \Lambda^2(V^*)$, prove that ω is symplectic if and only if $\omega^k \neq 0$.

9.2 Orientability of the Tangent Bundle

Exercise 9.2. Let M be a differentiable manifold of dimension n. Prove that the tangent bundle TM, considered as a $2n$-dimensional smooth manifold, is always orientable.

9.3 Lie Derivatives and Cartan's Formula

Exercise 9.3. Let M be a smooth manifold, α, β differential forms and X a vector field on M.

(1) Prove the following formula:
$$L_X(\alpha \wedge \beta) = (L_X\alpha) \wedge \beta + \alpha \wedge (L_X\beta)$$

(2) Deduce the Cartan formula
$$L_X\alpha = di_X\alpha + i_X d\alpha$$
e.g. by induction over the degree of α.

9.4 Integrability of Distributions and the Frobenius Criterion

Exercise 9.4. Let M be a smooth manifold and $\alpha \in \Omega^1(M)$ a nowhere vanishing 1-form. Define a vector bundle $E = \ker\alpha \subseteq TM$ with fibres spanned by the values of vector fields in the kernel of α. Show that E is integrable if and only if $\alpha \wedge d\alpha = 0$.

Problem Sheet 10

Integration, Orientability, and Topological Applications

10.1 Orientability and Non-Orientable Manifolds

Exercise 10.1. For an oriented manifold M, a smooth map $f : M \to M$ is called orientation preserving if $df_x : T_xM \to T_{f(x)}M$ is orientation preserving for all $x \in M$.

(1) Let $f : S^2 \to S^2$, $x \mapsto -x$ be the antipodal map. Show that f is not orientation-preserving on S^2.

(2) Prove that $\mathbb{RP}^2$ is not orientable.

10.2 Stokes' Theorem and the Brouwer Fixed-Point Theorem

Exercise 10.2.

(1) Let M be a compact oriented differentiable manifold with boundary $\partial M \neq 0$. Use Stokes' Theorem to show that there is no retraction of M onto ∂M, i.e. no smooth map $r : M \to \partial M$ s.t. $r\Big|_{\partial M} = \mathrm{Id}_{\partial M}$.

(2) Deduce the Brouwer fixed point theorem: Every smooth map $f : B^n \to B^n$, where $B^n := \{x \in \mathbb{R}^n \mid \|x\| < 1\}$ is the open unit ball, has a fixed point.

10.3 Homotopy Invariance of Integrals of Closed Forms

Exercise 10.3. Let M, N, be differentiable manifolds without boundary of dimensions m, n. Two smooth maps $f, g : M \to N$ are called homotopic if there exists a smooth map $H : M \times [0,1] \to N$ such that $H(-,0) = f$ and $H(-,1) = g$. Suppose M is closed and orientable, $\omega \in \Omega^m(N)$ a closed m-form on N and f, g homotopic. Show that

$$\int_M f^*\omega = \int_M g^*\omega$$

10.4 Exact 1-Forms and Integration over Loops

Exercise 10.4. Let M be a connected differentiable manifold and $\alpha \in \Omega^1(M)$ a 1-form. For a smooth map $\phi : S^1 \to M$ define

$$\int_\phi \alpha := \int_{S^1} \phi^*\alpha$$

Show that α is exact if and only if $\displaystyle\int_\phi \alpha = 0$ for all $\phi : S^1 \to M$.

Problem Sheet 11

Vector Bundles, Connections, and Topological Invariants

11.1 Homotopy Invariance and de Rham Cohomology

Exercise 11.1.

(1) Let M be a compact oriented differentiable manifold without boundary and $p \in M$ a point. Show that the constant map $f : M \to \{p\} \subseteq M$ is not homotopic to the identity map $M \to M$.

(2) Let $\phi_n : S^1 \to S^1, z \mapsto z^n$, where $S^1 = \{z \in \mathbb{C} \mid \|z\| = 1\}$ and let $[\omega] \in H^1_{dR}\left(S^1\right)$ be the class of a volume form on S^1. Show that $\phi_n^*[\omega] = n \cdot [\omega]$ and deduce that the maps ϕ_n are pairwise non-homotopic.

11.2 Flat Connections and Their Existence

Exercise 11.2. A connection on a vector bundle is called flat if it has vanishing curvature.

(1) Show that every trivial vector bundle over a differentiable manifold admits a flat connection.

(2) Find a vector bundle over $T^2 = S^1 \times S^1$ which is non-trivial but admits a flat connection.

11.3 Connections Induced by Embeddings

Exercise 11.3. Let $M \subseteq \mathbb{R}^n$ be a submanifold. Via this inclusion, we consider TM as a subbundle of $T\mathbb{R}^n\Big|_M$. Consider the splitting $T\mathbb{R}^n\Big|_M = TM \oplus TM^\perp$, where $TM^\perp$ denotes the orthogonal complement with respect to the standard scalar product on $\mathbb{R}^n$. Use this splitting and a flat connection as in the previous exercise to define a connection on TM.

11.4 Induced Connections on Associated Bundles

Exercise 11.4. Let $E \to M$ and $F \to M$ be two vector bundles with connections ∇_E and ∇_F. Find definitions for induced connections on the dual bundle E^*, the tensor product bundle $E \otimes F$ and the homomorphism bundle $\mathrm{Hom}(E, F)$. Verify that your definitions indeed define connections.

Problem Sheet 12

Vector Bundle Connections: Curvature and Parallel Section

12.1 Curvature Transformation Under Frame Changes

Exercise 12.1. Let $E \to M$ be a rank k vector bundle with connection ∇ and $\{s_i\}_{i=1}^k$, $\{s'_i\}_{i=1}^k$ local frames over an open set U. Define the matrix g by $s' = g \cdot s$, i.e. $s'_i(p) = \sum_{j=1}^{k} g_{ij}(p) s_j(p)$ in every $p \in U$. Show that the matrices of curvature 2-forms are related by $\Omega' = g \cdot \Omega \cdot g^{-1}$, i.e.

$$\Omega'_{ij} = \sum_{l,m=1}^{k} g_{il} \cdot \Omega_{lm} \cdot \left(g^{-1}\right)_{mj}.$$

12.2 Curvature in Induced Connections

Exercise 12.2. Let $V \to M$ and $W \to M$ be two vector bundles with connections ∇_V and ∇_W with curvatures F^{∇_V} and F^{∇_W}. Express the curvature of the induced connections (as in Exercise 11.4) on the dual bundle V^*, the tensor product bundle $V \otimes W$ and the homomorphism bundle $\mathrm{Hom}(V, W)$ in terms of F^{∇_V} and F^{∇_W}.

12.3 Curvature via Covariant Derivatives

Exercise 12.3. Let $E \to M$ be a vector bundle with connection ∇ with curvature $F^\nabla \in \Omega^2(\mathrm{End}\, E)$. Show that

$$F^\nabla(X, Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s,$$

for all vector fields X, Y on M and all sections s of E.
(Hint: Show that both sides of the equation are $C^\infty(M)$-linear in X, Y, s.)

12.4 Parallel Sections and Their Properties

Exercise 12.4. Let M be a connected differentiable manifold and $E \to M$ a vector bundle with connection ∇.

(1) Show that a parallel section has a zero if and only if it vanishes identically.

(2) Show that the set of parallel sections forms a finite-dimensional vector space.

Problem Sheet 13

Connections on Vector Bundles: Equivalence and Curvature

13.1 Integrable Subbundles and Torsion-Free Connections

Exercise 13.1. Let M be a smooth manifold and $U \subseteq TM$ a subbundle. Given a connection ∇ on TM, we say U is parallel (with respect to ∇) if $\nabla_X Y \in \Gamma(U)$ for all $X \in \mathfrak{X}(M)$ and $Y \in \Gamma(U)$. Prove that there exists a torsion free connection on TM such that U is parallel if and only if U is integrable.
(Hint: You may use that TM always admits a torsion-free connection. This has been proved in the script.)

13.2 Non-Commutativity of Connections

Exercise 13.2. Show that no connection on a positive-dimensional manifold M can satisfy $\nabla_X Y = \nabla_Y X$ for all $X, Y \in \mathfrak{X}(M)$.

13.3 Covariant Derivative and Parallel Transport

Exercise 13.3. Let M be a smooth manifold and $E \to M$ a vector bundle with connection ∇. Let $s \in \Gamma(E)$, $X \in \mathfrak{X}(M)$ and $p \in M$ a point. Suppose $c : I \to M$ is an integral curve of X with $c(0) = p$. Prove that

$$(\nabla_X s)(p) = \frac{d}{dt}(P_t^{-1}\left(s_{c(t)}\right)\bigg|_{t=0}.$$

where $P_t : E_p \to E_{c(t)}$ is the parallel transport along c from 0 to t.

13.4 Gauge Equivalence and de Rham Cohomology

Exercise 13.4. Let M be a smooth manifold and $E = M \times \mathbb{R} \to M$ the trivial rank 1 vector bundle. Two connections ∇, ∇' are called gauge-equivalent if there exists a smooth map $g : M \to \mathbb{R}^*$ such that $\nabla' s = g^{-1} \cdot \nabla(g \cdot s)$ for all sections s of E. Let ∇ be a flat connection. Then, in the given trivialization of E, the connection 1-form $\alpha := \nabla - d \in \Omega^1(\mathrm{End}(E)) = \Omega^1(M)$ is closed. Conversely, every closed 1-form α on M determines a flat connection via $\nabla^\alpha := d + \alpha$. Show that

$$\nabla^\alpha, \nabla^{\alpha'} \text{ are gauge-equivalent} \Leftrightarrow [\alpha] = [\alpha'] \in H^1_{dR}(M).$$

Problem Sheet 14

Riemannian Vector Bundles and Curvature

14.1 Symmetric and Skew-Symmetric Endomorphism Decomposition

Exercise 14.1. Let $(V, \langle\ ,\ \rangle)$ be a finite-dimensional vector space with scalar product. For $\phi \in \mathrm{End}(V)$ define its adjoint ϕ^* via $\langle \phi^* v, v' \rangle = \langle v, \phi v' \rangle$ for all $v, v' \in V$. The endomorphism ϕ is called symmetric if $\phi^* = \phi$ and skew-symmetric if $\phi^* = -\phi$.

(1) Prove that $(\phi^*)^* = \phi$ and that there is a direct sum decomposition $\mathrm{End}(V) = V_+ \oplus V_-$, where V_+, resp. V_-denote the vector spaces of symmetric, resp. skew-symmetric endomorphisms.

(2) Let $E \to B$ be a vector bundle with a metric $\langle\ ,\ \rangle$. Do the same construction as above fibrewise for the fibres of $\mathrm{End}(E)$ to deduce that there are vector bundles E_+, E_- with $\mathrm{End}(E) = E_+ \oplus E_-$ and that the sections of E_- are the skew-symmetric endomorphisms of E as defined in the script.

14.2 Skew-Symmetric Endomorphisms and Exterior Algebra

Exercise 14.2. Let $E \to B$ be a vector bundle with a metric $\langle\ ,\ \rangle$. We use the same notation as above. Show that $E_- \cong \Lambda^2 E$ and conclude that if E is oriented and of rank 2, then $E_-$is the trivial bundle.

14.3 Uniqueness of the Riemann Curvature Tensor

Exercise 14.3. Let $(V, \langle\ ,\ \rangle)$ be a vector space with a scalar product. Let $R, R' : V \times V \times V \to V$ be two trilinear functions satisfying the symmetries of the Riemann curvature tensor. Assume that for all $X, Y, Z \in V$ one has $\langle R(X,Y)Y, Z \rangle = \langle R'(X,Y)Y, Z \rangle$. Show that $R = R'$.

14.4 Product Metrics and Curvature Properties

Exercise 14.4. Let $(M, \langle\ ,\ \rangle_M)$ and $(N, \langle\ ,\ \rangle_N)$ be Riemannian manifolds. Consider the product metric on $M \times N$, defined in every tangent space $T_{(a,b)} M \times N = T_a M \oplus T_b M$ via $\langle X_M + X_N, Y_M + Y_N \rangle_{M \times N} = \langle X_M, Y_M \rangle_M + \langle X_N, Y_N \rangle_N$.

(1) Express the Levi-Civita connection of the product metric on $M \times N$ in terms of the Levi-Civita connections of the factors.

(2) Show that the sectional curvature is zero on any plane in $T_{(a,b)}(M \times N)$ which is spanned by one vector in $T_a M$ and one in $T_b N$.