Differentiable Manifolds

Xumin Liang

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Preface

This script is mainly based on Prof. Dr. Dieter Kotschick's course on Differential Geometry for Ludwig-Maximilians-Universität in Munich in winter semester 2023-2024.

Motivation and Scope

Differentiable manifolds provide a unified framework for studying spaces that locally resemble Euclidean space but may exhibit complex global behavior. From the curvature of spacetime in general relativity to the configuration spaces of mechanical systems, manifolds lie at the heart of many physical and mathematical phenomena. This text focuses on developing the core concepts of smooth manifolds, tangent spaces, vector bundles, and differential forms—tools essential for advanced topics such as Lie theory, Riemannian geometry, and cohomology.

While the material is rooted in pure mathematics, the techniques presented here have profound applications in theoretical physics, including gauge theory, symplectic mechanics, and string theory. Our goal is not merely to enumerate definitions and theorems but to cultivate an intuitive grasp of the subject through carefully chosen examples, historical context, and connections to adjacent fields.

Structure and Pedagogy

The book is organized into 14 chapters, progressing from foundational material to advanced topics. Key pedagogical features include:

- **Gradual Complexity**:
  - Chapters 1–2 introduce topological and differentiable manifolds, emphasizing local coordinates, atlases, and the "smooth invariance of domain."
  - Chapters 3–6 explore tangent spaces, vector bundles, and their geometric operations (e.g., pullbacks, metrics, and subbundles).
  - Chapters 7–9 delve into dynamical systems (flows), Lie theory, and the Frobenius theorem.
  - Chapters 10–14 culminate in differential forms, integration, de Rham cohomology, and connections.
- **Examples and Theorems**:

  - Classical examples (e.g., spheres, tori, projective spaces) recur throughout the text.
  - Major theorems—such as **Whitney's Embedding Theorem**, **Sard's Theorem**, and **Stokes' Theorem**—are presented with detailed proofs.

- **Visual and Algebraic Balance**:
  - Geometric intuition is prioritized through diagrams while maintaining algebraic rigor.
  - Exercises interspersed within chapters encourage active learning.

Prerequisites and Approach

Readers should be familiar with:

- Basic topology (open/closed sets, compactness, Hausdorff spaces),
- Linear algebra (vector spaces, dual spaces, multilinear maps),
- Calculus on Euclidean spaces (partial derivatives, inverse function theorem).

Abstract definitions (e.g., vector bundles, differential forms) are motivated by their classical analogs in $\mathbb{R}^n$. For instance:

- **Tangent spaces** generalize directional derivatives,
- **Vector bundles** formalize parameterized vector spaces,
- **Differential forms** unify integration and differentiation.

Philosophy and Innovations

Three principles guide this work:

- **Accessibility**: Technical machinery (e.g., partitions of unity) is introduced only when necessary.
- **Interconnectedness**: Concepts reappear in new contexts (e.g., the tangent bundle underpins flows and Lie derivatives).
- **Modern Relevance**: Applications are hinted at throughout (e.g., the Frobenius theorem foreshadows foliations).

Acknowledgments

This manuscript owes its existence to countless conversations with colleagues, students, and mentors. Special thanks to the vibrant mathematical community for their insights and encouragement. Feedback from readers is warmly welcomed.

To the Reader

"The questions are the breath of research,"

——Hermann Weyl

Differential geometry is a journey—one that begins with coordinates and curves and leads to the frontiers of modern physics. While the path is challenging, the rewards are profound. Approach each chapter with patience, revisit examples often, and let curiosity guide you.

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Contents

Chapter 1

Topology

1.1 Topological Space

Definition 1.1. A **topological space** $(X, \mathcal{O})$ is a set together with open sets $\mathcal{O} \subset \mathcal{P}(X)$, s.t.

(1) $\varnothing$, $X \in \mathcal{O}$;

(2) $U_1, U_2 \in \mathcal{O} \Rightarrow U_1 \cap U_2 \in \mathcal{O}$;

(3) $U_i \in \mathcal{O}$, $i \in I \Rightarrow \bigcup\limits_{i \in I} U_i \in \mathcal{O}$.

Example 1.1.

(1) $\mathcal{O} = \{\varnothing, X\}$ the **trivial topology** on X.

(2) $\mathcal{O} = \mathcal{P}(X)$ the **discrete topology**.

(3) the **metric topology** on a metric space.

1.2 Metric Spaces

Definition 1.2. A **metric space** (X, d) is a set X together with

$$d : X \times X \to \mathbb{R}(x, y) \quad \mapsto d(x, y)$$

s.t.

(1) $d(x, y) \geqslant 0$ with "=" if and only if $x = y$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, z) \leqslant d(x, y) + d(y, z)$, $\forall x, y, z \in X$.

In the metric topology, a subset $U \subset X$ is open if $\forall x \in U$, $\exists \varepsilon > 0$, s.t.

$$B(x, \varepsilon) := \{y \in X \mid d(x, y) < \varepsilon\} \subset U.$$

Terminology. Let $(X, \mathcal{O})$ be a topological space.

(1) $V \subset X$ is **closed** if $X \setminus V \in \mathcal{O}$.

(2) $x \in X$, $W \subset X$ is a **neighborhood** of x in $(X, \mathcal{O})$, if $x \in W$ and W contains an open set U, s.t. $x \in U \subset W$.

(3) $U_i \in \mathcal{O}$, $i \in I$, the U_i form an **open cover** of X if $\bigcup_{i \in I} U_i = X$.

Definition 1.3. A topological space $(X, \mathcal{O})$ is **Hausdorff** if $\forall x_1, x_2 \in X$, $x_1 \neq x_2$, $\exists U_1, U_2 \in \mathcal{O}$, s.t. $x_i \in U_i$ and $U_1 \cap U_2 = \varnothing$.

Example 1.2. The metric topology of a metric space is always Hausdorff.

Proof. Let $x, y \in X$ and $x \neq y$. Then $d(x, y) > 0$. Take $\varepsilon := \dfrac{d(x,y)}{2}$, then $B(x, \varepsilon) \cap B(y, \varepsilon) = \varnothing$ and $x \in B(x, \varepsilon)$, $y \in B(y, \varepsilon)$. $\square$

1.3 Basis of Topology

Definition 1.4. A **basis** of the topology $\mathcal{O}$ is a $B \subset \mathcal{P}(X)$, s.t. every $U \in \mathcal{O}$ is a union of subsets in B.

Lemma 1.1. Consider $\mathbb{R}^n$ with the metric topology induced by Euclidean distance function

$$d(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{\frac{1}{2}}$$

There is a countable basis $B \subset \mathcal{P}(\mathbb{R}^n)$.

Proof. Take $B\left(x, \dfrac{1}{k}\right)$, where $x \in \mathbb{Q}^n$, $k \in \mathbb{N}$. $\mathcal{B}$ consists of all these balls as x ranges over $\mathbb{Q}^n$ and k ranges over $\mathbb{N}$.

$U \subset \mathbb{R}^n$ open. Take $x \in U$. Then $\exists \varepsilon > 0$, s.t. $B(x, \varepsilon) \subset U$. Take $y \in B\left(x, \dfrac{1}{3}\varepsilon\right) \cap \mathbb{Q}^n$.

Consider $x \in B\left(y, \dfrac{2}{3}\varepsilon\right) \subset U$.

$$d(x, y) < \frac{1}{3}\varepsilon$$

Fix $r \in \mathbb{Q}$ with $\dfrac{1}{3}\varepsilon < r < \dfrac{2}{3}\varepsilon$. Then $B(y, r) \in \mathcal{B}$ and $B(y, r) \subset U$. $\square$

1.4 Topological Manifold

Definition 1.5. A **topological manifold** M of dimension $n \in \mathbb{N}$ is a topological space $(M, \mathcal{O})$, s.t.

(1) $(M, \mathcal{O})$ is locally homeomorphic to $\mathbb{R}^n$ ("locally Euclidean");

(2) $(M, \mathcal{O})$ is Hausdorff;

(3) $(M, \mathcal{O})$ has a countable basis for $\mathcal{O}$.

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces.

Definition 1.6. A map $f : X \to Y$ is **continuous** if $f^{-1}(U) \in \mathcal{O}_X$ for all $U \in \mathcal{O}_Y$.

Definition 1.7. f is **homeomorphism** if f is bijective and continuous, and f^{-1} is also continuous.

Definition 1.8. $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are locally homeomorphic if every $x \in X$ has an open neighborhood U which is homeomorphic to an open set in Y.

Example 1.3.

(1) $M = \mathbb{R}^n$.

(2) M is a manifold $\Rightarrow$ any open $U \subset M$ is also a manifold.

(3) M is a manifold of dimension m and N is a manifold of dimension $n \Rightarrow M \times N$ is a manifold of dimension $m + n$.

(4) $S^n := \{x \in \mathbb{R}^{n+1} \mid ||x|| = 1\}$. This is a n-dimensional manifold.

(5) $T^n = \underbrace{S^1 \times \cdots \times S^1}_{n \text{ times}}$ by (3) and (4).

(6) Every surface is a 2-dimensional manifold.

Chapter 2

Differentiable Manifold

2.1 Charts

Locally Euclidean: $\forall x \in X$, $\exists U$ open and a homeomorphism $\varphi : U \to V \subset \mathbb{R}^n$.

Define (U_1, φ_1) and (U_2, φ_2) as above. Then

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \xrightarrow{\text{homeomorphism}} \varphi_2(U_1 \cap U_2).$$

The (U_i, φ_i) are called **charts** and $f_{21} = \varphi_2 \circ \varphi_1^{-1}$ is the transition map from the chart (U_1, φ_1) to the chart (U_2, φ_2).

2.2 Atlas

Definition 2.1. A collection of charts (U_i, φ_i), $i \in I$ with $\bigcup\limits_{i\in I} U_i = M$ is called an **atlas**. We have the **cocycle conditions/properties**

$$\begin{aligned} &(1)\ f_{ii} = \mathrm{Id} \\ &(2)\ f_{ij} = f_{ji}^{-1} \qquad\quad \forall i, j, k \in I \\ &(3)\ f_{ij} f_{jk} = f_{ik} \end{aligned}$$

The f_{ij} for pairs $i, j \in I$ with $U_i \cap U_j \neq \varnothing$ form the structure cocycle of the given atlas

$$\mathscr{A} = \{(U_i, \varphi_i) \mid i \in I\}.$$

Proposition 2.1. Let $\mathscr{A}$ be an atlas for M. From the collection of open subsets $V_i = \varphi_i(U_i) \subset \mathbb{R}^n$ together with the structure cocycle, one can reconstruct M.

Proof. $\overline{M} = \left(\coprod\limits_{i\in I} V_i \right) / \sim$, where $\sim$ is the equivalence relation given by $V_i \ni p \sim q = f_{ji}(p) \in V_j$, $\forall i, j \in I$.

$$\begin{aligned} a : \overline{M} &\to M \\ [p] &\mapsto \varphi_i^{-1}(p) \qquad \text{if } p \in V_i \end{aligned}$$

If $q \in V_j$ is equivalent to p, then $q = f_{ji}(p) \Rightarrow \varphi_j^{-1}(q) = \varphi_j^{-1}(\varphi_j^{-1}\varphi_i^{-1})(p) = \varphi_i^{-1}(p)$. So a is well-defined. a is also continuous.

$$\begin{aligned} b : M &\to \overline{M} \\ m &\mapsto [\varphi_i(m)] \qquad \text{if } m \in U_i \end{aligned}$$

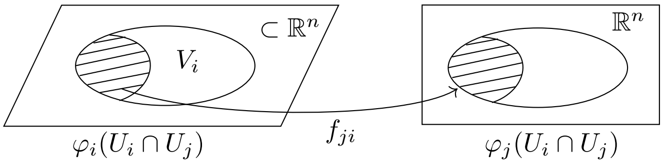
If m is also in U_j, then $\varphi_j(m) = (\varphi_j \circ \varphi_i^{-1})\varphi_i(m) = f_{ji}(\varphi_i(m))$. So b is well-defined. $b\big|_{U_i} = \pi \circ \varphi_i$, where $\pi : \coprod_{i \in I} V_i \to M$ is the projection onto equivalent classes. Thus b is continuous.

$$\begin{array}{ccccccc} \overline{M} & \overset{a}{\to} & M & \overset{b}{\to} & \overline{M} & & \\ [p] & \mapsto & \varphi_i^{-1}(p) & \mapsto & [\varphi_i\varphi_i^{-1}(p)] = [p] & \Rightarrow & b \circ a = \mathrm{Id}_{\overline{M}} \\ M & \overset{b}{\to} & \overline{M} & \overset{a}{\to} & M & & \\ m & \mapsto & [\varphi_i(m)] & \mapsto & \varphi_i^{-1}\varphi_i(m) = m & \Rightarrow & a \circ b = \mathrm{Id}_M \end{array}$$

□

2.3 Differentiable Manifold

Definition 2.2. A **smooth or differentiable manifold** is a topological manifold together with an atlas $\mathscr{A}$ for which f_{ij} are smooth/differentiable.



Smooth means $\mathcal{C}^r$ for some $r \geqslant 2$.

Terminology. Such an atlas is called a **smooth atlas**. Two smooth atlases $\mathscr{A}_1 = \{(U_i, \varphi_i) \mid i \in I\}$ on M are **equivalent** if $\mathscr{A}_1 \cup \mathscr{A}_2$ is also a smooth atlas.
$\mathscr{A}_2 = \{(U'_k, \varphi'_k) \mid k \in I'\}$

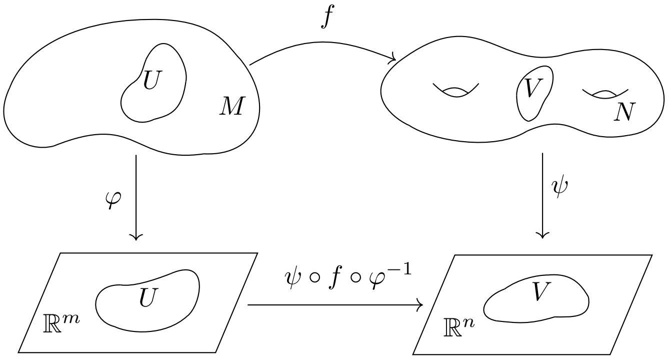
2.4 Differentiable Structure

Definition 2.3. A **differentiable structure** on M is a maximal smooth atlas, equivalently an equivalence class of atlases for the above.

Fact. Every maximal $\mathcal{C}^r$ atlas contains a unique maximal $\mathcal{C}^\infty$ atlas. Because of this, we will only consider $\mathcal{C}^\infty$ manifolds.

$$\text{smooth} = \text{differentiable} = \mathcal{C}^\infty$$

Definition 2.4. Let M and N be smooth manifolds, $f : M \to N$ is **smooth** if $\forall p \in M$, $\exists$ a chart (U, φ) with $p \in U$ and a chart (V, ψ) for N with $f(p) \in V$ such that $\psi \circ f \circ \varphi^{-1}$ is smooth.



Example 2.1. $f : M \to \mathbb{R}$ is smooth if and only if $f \circ \varphi^{-1}$ is smooth for all charts (U, φ).

Definition 2.5. $f : M \to N$ is a **diffeomorphism** if it is bijective, differentiable, and f^{-1} is also differentiable.

Example 2.2. Every $B(x, \varepsilon) \subset \mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^n$.

Remark. Not every topological manifold has a differentiable structure. If it has one, it may fail to be unique!

For $n \leqslant 3$, every topological manifold has a differentiable structure, unique up to diffeomorphism.

For $n \geqslant 4$, there are manifolds with no differentiable structure, and there are manifold with unusual non-diffeomorphic differentiable structures.

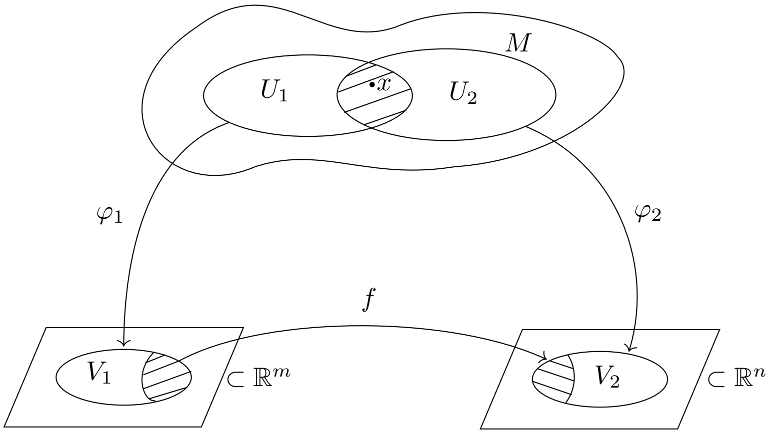
Example 2.3. The topological manifold $\mathbb{R}^4$ has infinitely many distinct differentiable structures.

Example 2.4. S^7 has several distinct differentiable structures.

2.5 “The Smooth Invariance of Domain”

Differentiable atlas means that transition functions between charts are diffeomorphisms. The way we had defined differentiable manifolds, we assume that we always have a fixed dimension, so we define a manifold of dimension n which is locally homeomorphic to $\mathbb{R}^n$. Now we want to show that in the differentiable case, functions as dimension given are actually redundant.

Take a manifold M. Assume we have two charts U_1, U_2, and $\varphi_1 : U_1 \to V_1 \subset \mathbb{R}^m$ and $\varphi_2 : U_2 \to V_2 \subset \mathbb{R}^n$.



Then we have a transition map $f_{21} = \varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$.

If the transition map f_{12}, f_{21} are diffeomorphisms, then $m = n$. Since

$$\begin{array}{lcl} f_{12} \circ f_{21} = \mathrm{Id}_{\varphi_1(U_1 \cap U_2)} & \overset{\text{differentiate}}{\rightsquigarrow} & D_{\varphi_2(x)} f_{12} \circ D_{\varphi_1(x)} f_{21} = \mathrm{Id}_{\mathbb{R}^m} \\ f_{21} \circ f_{12} = \mathrm{Id}_{\varphi_2(U_1 \cap U_2)} & & D_{\varphi_1(x)} f_{21} \circ D_{\varphi_2(x)} f_{12} = \mathrm{Id}_{\mathbb{R}^n} \end{array}$$

Both derivatives on the LHS are isomorphisms.

$$\begin{tikzcd} \mathbb{R}^m \arrow[r, shift left, "D_{\varphi_1(x)} f_{21}"] & \mathbb{R}^n \arrow[l, shift left, "D_{\varphi_2(x)} f_{12}"] \end{tikzcd}$$

which implies

$$m = n$$

This is called "the smooth invariance of domain".

Given a smooth manifold M with a smooth atlas (U_i, φ_i), $i \in I$, we can reconstruct M up to diffeomorphism just from $\varphi_i(U_i)$, $i \in I$, together with the structure cocycle given by the transition function f_{ij}.

Chapter 3

Tangent Spaces and Tangent Bundle

Let M be a smooth manifold and $\mathscr{A} = \{(U_i, \varphi_i) \mid i \in I\}$ a differentiable atlas. All the φ_i take values in $\mathbb{R}^m$, $n = \dim M$. Consider triples $(x, i, v) \in M \times I \times \mathbb{R}^n$ with $x \in U_i$. On the set of such triples define the relation $(x, i, v) \sim (y, j, w)$ by $x = y$ and $D_{\varphi_i(x)} \underbrace{(\varphi_j \circ \varphi_i^{-1})}_{f_{ji}}(v) = w$. Then

$$(D_{\varphi_j(y)} f_{ij})(w) = v.$$

Claim 3.1. This is an equivalence relation.

$(x, i, v) \sim (y, j, w) \sim (z, k, t)$
$x = y = z$

$$\underbrace{D_{\varphi_j(x)} f_{kj} \circ D_{\varphi_i(x)} f_{ji}}_{D_{\varphi_i(x)} f_{ki}}(v) = (D_{\varphi_j(x)} f_{kj}) w = t$$

Let TM be the set of equivalence classes, and

$$\begin{aligned} \pi : TM &\to M \\ [x, i, v] &\mapsto x \end{aligned}$$

If $A \subset M$, then $\pi^{-1}(A) = T_A M$.
If $A = \{x\}$, then $\pi^{-1}(x) = T_x M$, the tangent space to M at x.
If $A \subset M$ is open, then A is itself a manifold, and $TA = T_A M$.
For every chart (U_i, φ_i), we have a bijective map

$$\begin{aligned} T\varphi_i : TU_i &\to \varphi_i(U_i) \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n \\ [x, i, v] &\mapsto (\varphi_i(x), v) \end{aligned}$$

$$\begin{tikzcd}
T(U_i \cap U_j) \arrow[r, "T\varphi_i"] \arrow[d, "T\varphi_j"'] & \varphi_i(U_i \cap U_j) \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n} \arrow[dl, "T\varphi_j \circ (T\varphi_i)^{-1}(z, v) = (f_{ji}(z), \underbrace{D_z f_{ji}(v)}_{w})"'] \\
\varphi_j(U_i \cap U_j) \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n &
\end{tikzcd}$$

We give each TU_i the unique topology which makes $T\varphi_i$ into a homeomorphism. This is well-defined. On TM, we define topology by requiring each TU_i to be open, and itself have the topology defined via $T\varphi_i$.

We consider $\mathscr{A}' = \{(TU_i, T\varphi_i) \mid i \in I\}$ as an atlas for TM. This has $\mathcal{C}^\infty$ transition maps, and so values TM into a $\mathcal{C}^\infty$ manifold.

With respect to this differentiable structure on TM, the projection $\pi : TM \to M$ is a differentiable map.

Lemma 3.2. For every $x \in M$, the tangent space T_xM has a well-defined structure as a $\mathbb{R}$-vector space of $\dim n$.

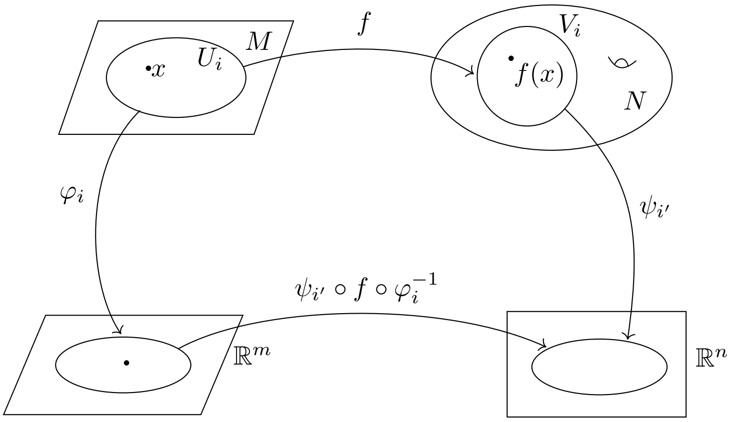
Proof. Suppose $x \in U_i$, then $T\varphi_i\Big|_{T_xM} : T_xM \to \{\varphi_i(x)\} \times \mathbb{R}^n$ is bijective. Define the vector space structure on T_xM to be the unique one that makes $T\varphi_i\Big|_{T_xM}$ a linear isomorphism. If $x \in U_j$, then $f_{ji} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is a diffeomorphism. The derivative is linear

$$D_{\varphi_i(x)} f_{ji} : \mathbb{R}^n \to \mathbb{R}^n.$$

This is an isomorphism of the vector space. This shows that the vector space structure on T_xM defined using (U_j, φ_j) instead of (U_i, φ_i) is isomorphic to the one gotten from U_i. ☐

For every $x \in M$, $\pi^{-1}(x) = T_xM$ is a vector space.

Suppose $f : M \to N$ is a differentiable map between differentiable manifolds.



Define $Df : TM \to TN$

$$[x, i, v] \mapsto [f(x), i, D_{\varphi_i(x)}(\psi_{i'} \circ f \circ \varphi_i^{-1})(v)]$$

Suppose (U_j, φ_j) is another chart for M with $x \in U_j$.

$$[x, i, v] = [x, , D_{\varphi_i(x)} f_{ji}(v)] \mapsto [f(x), i', D_{\varphi_j(x)}(\psi_{i'} \circ f \circ \varphi_j^{-1}) D_{\varphi_i(x)} f_{ji}(v)]$$
$$(\psi \circ f \circ \varphi_j^{-1}) \circ f_{ji} = (\psi \circ f \circ \varphi_j^{-1}) \circ (\varphi_j \circ \varphi_i^{-1}) = \psi \circ f \circ \varphi_i^{-1}$$
$$D_{\varphi_j(x)}(\psi \circ f \circ \varphi_j^{-1}) \circ D_{\varphi_i(x)} f_{ji} = D_{\varphi_i(x)}(\psi \circ f \circ \varphi_i^{-1})$$

In the same way, one checks that Df does **not** depend on the chart used for N.

$Df\Big|_{T_xM} : T_xM \to T_{f(x)}N \subset TN$ is a linear map between tangent spaces.

$$[x, i, v] \mapsto [f(x), i', D_{\varphi_i(x)}(\psi_{i'} \circ f \circ \varphi_i^{-1})(v)]$$

Definition 3.1. $D_x f := Df\Big|_{T_x M}$ is the **derivative** of f at $x \in M$.

Chapter 4

Paracompactness

4.1 Compact and Paracompact

Definition 4.1. A topological space $(x, \mathcal{O})$ is **compact** if every open covering has a finite subcover.

Example 4.1.

- Compact $\{x\}$, $[0,1]$, S^1, S^n, T^n.
- Not compact $(0,1)$, $(0,1]$, $\mathbb{R}$, $\mathbb{R}^n$.

Definition 4.2. A topological space $(X, \mathcal{O})$ is **paracompact** if every open covering has a locally finite refinement.

Definition 4.3. Let $\{U_i \mid i \in I\}$, be a collection of subsets in X. This collection is **locally finite** if $\forall x \in X$, there exists an open neighborhood U_x, s.t. $U_i \cap U_x \neq \varnothing$ for only finitely many $i \in I$.

Definition 4.4. Let U_i, $i \in I$ be a covering of X, i.e. $\bigcup\limits_{i \in I} U_i = X$. A **refinement** of this covering is a covering by subsets V_k, $k \in K$, such that $\forall k \in K$, $\exists i = i(k) \in I$, s.t. $V_k \subset U_i$.

Proposition 4.1. Let $\{U_i \mid i \in I\}$ be an open covering of a manifold M. There exists an atlas $\mathscr{A} = \{(V_k, \varphi_k) \mid k \in K\}$ such that

(1) $\varphi_k(V_k) = B(x_k, 3) \subset \mathbb{R}^n$;

(2) $W_k = \varphi_k^{-1}(B(x_k, 1))$ form a covering of M;

(3) The V_k form a locally finite refinement of the covering by the U_i.

Proof. Step 1: There exists a sequence G_i, $i = 1, 2, \ldots$ of open subsets on M with $\overline{G}_i \subset G_{i+1}$ $\forall i$, $\overline{G}_i$ compact $\forall i$, and $\bigcup\limits_{i=1}^{\infty} G_i = M$.

The topology of M has a countable basis consisting of open sets A_j, $j = 1, 2, \ldots$, with compact closures G.

$G_1 = A_1$. Suppose G_k has been defined as $G_k = A_1 \cup \cdots \cup A_{j_k}$. Let j_{k+1} be the smallest natural number for which

$$\overline{G}_k \subset A_1 \cup \cdots \cup A_{j_{k+1}}$$

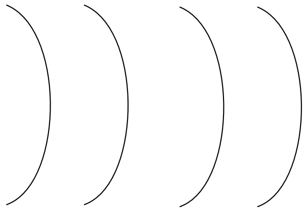
Define $G_{k+1} := A_1 \cup \cdots \cup A_{j_{k+1}}$. This sequence of G_k has all the properties required by Step 1.

Step 2: Given the open covering of M by the U_i, we can choose a chart (V_x, φ_x) for every $x \in M$, so that $x \in V_x$, $\varphi_x(V_x) = B(y_x, 3)$.

Let $W_x = \varphi_x^{-1}(B(y_x, 1))$. We may assume the V_x form a refinement of the U_i, i.e. $\forall x$, $\exists i = i(x)$, s.t. $V_x \subset U_i$.

Each set $\overline{G}_k \setminus G_{k-1}$ can be covered by finitely many such W_{x_i}, $i \in \{1, \ldots, l\}$, such that, moreover,

$$V_{x_i} \subset G_{k+1} \setminus \overline{G}_{k-2} = G_{k+1} \cap (M \setminus \overline{G}_{k-2}).$$



$$G_{k-2} \subset G_{k-1} \subset G_k \subset G_{k+1}$$

Now let $x \in M \Rightarrow m \in G_i$. Take $G_i \setminus \overline{G}_{i-1}$. This will intersect only finitely many V's. So V_{x_i} are a locally finite refinement. □

Example 4.2. Let M be compact. $\forall x \in M$, $\exists$ a chart of this form around x, $\{W_x \mid x \in M\}$ is an open covering of M. Because M is compact, $\exists x_1, \ldots, x_l \in M$, s.t. $\bigcup\limits_{i=1}^{l} W_{x_i} = M$.

4.2 Partition of Unity

Definition 4.5. Let M be a smooth manifold. Let $\{U_i \mid i \in I\}$ be an open covering of M. A **smooth partition of unity on M subordinate to the covering** $\{U_i \mid i \in I\}$ is a collection of smooth functions $\rho_j : M \to \mathbb{R}$ such that $\rho_j \geqslant 0$, $\forall j \in J$ for which the supports of the ρ_j form a locally finite refinement of the U_i and $\sum\limits_{j \in J} \rho_j \equiv 1$.

Definition 4.6. If $f : M \to \mathbb{R}$ is any continuous function, define $\operatorname{supp}(f) = \overline{\{x \in M \mid f(x) \neq 0\}}$.

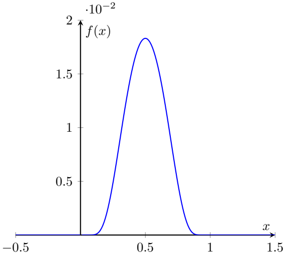
In the definition of partition of unity, we want $\operatorname{supp}(\rho_j) \subset U_{i(j)}$.

Theorem 4.2. If M is any smooth manifold and $\{U_i \mid i \in I\}$ is any open covering, then there is a subordinate smooth partition of unity.

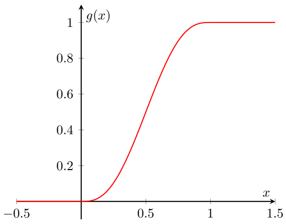
Proof. First consider the following smooth function $f : \mathbb{R} \to \mathbb{R}$.

$$f(x) = \begin{cases} \exp\left(\dfrac{1}{x(x-1)}\right) & \text{for } 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

is a $\mathcal{C}^\infty$ function, a **bump function**.



$$g(x) = \frac{\displaystyle\int_{-\infty}^{x} f(t)\ \mathrm{d}t}{\displaystyle\int_{\mathbb{R}} f(t)\ \mathrm{d}t}$$



is also $\mathcal{C}^\infty$.

Given the U_i, construct an atlas in the proof of the Proposition 4.1 $\{(V_k, \varphi_k) \mid k \in K\}$.

Define $\rho_k : M \to \mathbb{R}$ so that it is $\mathcal{C}^\infty$ and

$$\begin{aligned} \rho_k\Big|_{W_k} &\equiv 1 \\ \rho_k &\geqslant 0 \\ \mathrm{supp}(\rho_k) &\subset V_k \end{aligned}$$

The supports are thus a locally finite refinement of U_i and $s = \sum\limits_{k\in K} \rho_k$ is defined everywhere>0.

Define $\overline{\rho}_k := \dfrac{\rho_k}{s}$, $\sum\limits_{k\in K} \overline{\rho}_k \equiv 1$. □

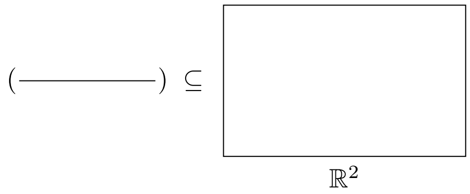
Chapter 5

Submanifold

5.1 Submanifold

Recall M is a smooth manifold and $U \subset M$ open $\Rightarrow U$ is a smooth manifold.

We want a broader definition of submanifold, e.g. incorporating things like $S^n \subset \mathbb{R}^{n+1}$ or



Definition 5.1. A subset $N \subset M$, where M is a smooth manifold called a **submanifold** if for every point $p \in N$, there exists a chart for M, centered at p, say (U, φ), such that

$$\varphi(N \cap U) = \{x_1 = \cdots = x_k = 0\} \cap \varphi(U) \subset \mathbb{R}^m$$

where $\dim M = m$ and k is some fixed non-negative integer.

Remark. Clearly, $\dim N = m - k \leqslant m = \dim M$.

5.2 Immersion, Submersion and Embedding

Definition 5.2. Let $f : M \to N$ be a map of smooth manifolds and $p \in M$.

(1) f is called an **immersion** at p, if $D_p f : T_p M \to T_{f(p)} N$ is injective.

(2) f is called a **submersion** at p, if $D_p f$ is surjective.

(3) f is called an **immersion/submersion**, if it is an immersion/submersion at all points in M.

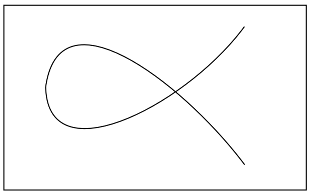
(4) f is called an **embedding**, if it is an immersion and a homeomorphism onto its image.

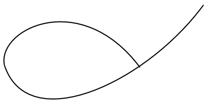
Example 5.1.

(1) $i : \mathbb{R}^m \to \mathbb{R}^n$ with $n \geqslant m$ is an immersion (and an embedding).
$(x_1, \dots, x_m) \mapsto (x_1, \dots, x_m, 0, \dots, 0)$

(2) $\pi : \mathbb{R}^m \to \mathbb{R}^n$ with $m \geqslant n$ is a submersion.
$(x_1, \dots, x_n, x_{n+1}, \dots, x_m) \mapsto (x_1, \dots, x_n)$

(3) $(a, b) \xrightarrow{\gamma} \mathbb{R}^2$ is an immersion but not an embedding.



Similarly,  is an immersion but not an embedding.

Remark. If $f : M \to N$ is an immersion, $\dim M \leqslant \dim N$. If $f : M \to N$ is a submersion, $\dim M \geqslant \dim N$.

Theorem 5.1. *Let $f : M \to N$ be an immersion at $p \in M$. Then, there exist charts (U, φ) around p and (V, ψ) around $q = f(p)$, s.t. $\psi \circ f \circ \varphi^{-1} = i\Big|_{\varphi(U)}$, i.e. $\psi \circ f \circ \varphi^{-1}(x_1, \dots, x_m) = (x_1, \dots, x_m, 0, \dots, 0)$.*

Proof. Take charts (U_0, φ_0) around p and (V_0, ψ_0) around q. The Jacobi matrix of $\psi_0 \circ f \circ \varphi_0^{-1}$ at 0 has rank $m = \dim M$ by assumption. After reordering the coordinates of ψ_0, we obtain a new chart (V_0, ψ), s.t. for $F = \psi \circ f \circ \varphi^{-1}$, $\left(\dfrac{\partial F_i}{\partial x_j}\right)_{\substack{i = 1, \dots, m \\ j = 1, \dots, m}}$ is invertible.

Now define $G : \varphi(U_0) \times \mathbb{R}^{n \times m} \to \mathbb{R}^n$

$$(x_1, \dots, x_m, x_{m+1}, \dots, x_n) \mapsto F(x_1, \dots, x_m) + (0, \dots, 0, x_{m+1}, \dots, x_n)$$

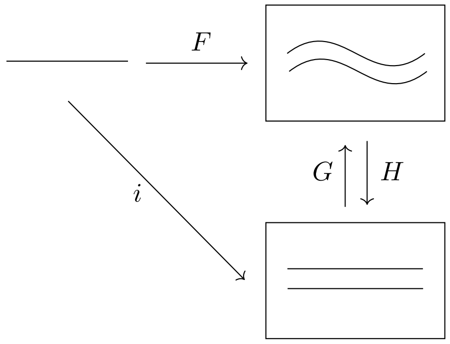
$D_0 G = \begin{pmatrix} \left(\dfrac{\partial F_i}{\partial x_j}\right)_{i,j=1}^{m} & * \\ 0 & \mathrm{Id} \end{pmatrix}$ is invertible. By the inverse function theorem, we find

$$0 \in \varphi(U) \underset{\text{open}}{\subseteq} \varphi(U_0) \qquad 0 \in \psi(V) \underset{\text{open}}{\subset} \psi(V_0)$$

and a smooth function H

$$H : \psi(V) \to \varphi(U) \times U_1$$

s.t. $G \circ H = \mathrm{Id}$ and $H \circ G = \mathrm{Id}$ where defined.



Now set $\tilde{\psi} = H \circ \psi$, then $\tilde{\psi} \circ f \circ \varphi_0^{-1} = H \circ F = H \circ G \circ i = i$. $\square$

Remark. We only needed to modify the chart for the target.

We also have

Theorem 5.2. If $f : M \to N$ is a submersion ($m \geqslant n$) at $p \in M$, there are charts (U, φ) around p and (V, ψ) around $q = f(p)$, s.t. $\psi \circ f \circ \varphi^{-1}(x_1, \dots, x_m) = (x_1, \dots, x_n)$.

Proof. Take arbitrary charts (V, ψ) and (U_0, φ_0) around q, respectively p. After reordering coordinates of φ_0, we may assume for $F = \psi \circ f \circ \varphi_0^{-1}$, we have $\left(\frac{\partial F_i}{\partial x_j}\right)_{i,j=1}^{n}$ is invertible. Define

$$G(x_1, \dots, x_n, x_{n+1}, \dots, x_m) = (F_1(x_1, \dots, x_m), \dots, F_n(x_1, \dots, x_m), x_{n+1}, \dots, x_m)$$

Then

$$D_0 G = \begin{pmatrix} \left(\frac{\partial F_i}{\partial x_j}\right)_{i,j=1}^{n} & * \\ 0 & \mathrm{Id} \end{pmatrix}$$

is invertible, so we have a local inverse H (possibly shrinking the domain of definition). Then set $\varphi = G \circ \varphi_0$ where defined. This gives

$$\psi \circ f \circ \varphi^{-1} = \psi \circ f \circ \varphi_0^{-1} \circ G^{-1} = F \circ H = \pi \circ G \circ H = \pi$$

$\square$

Theorem 5.3. Let $f : M \to N$ be an embedding. Then $f(M) \subset N$ is a submanifold.

Proof. Let $q \in f(M)$. Because f is a homeomorphism onto its image, there is a unique preimage p, s.t. $f(p) = q$ and a chart centered at q, say (V, ψ), s.t. $f^{-1}(V) = U \subset M$ admits a chart φ. Arguing as in the previous theorem, we can assume $(\psi \circ f \circ \varphi^{-1})(x_1, \dots, x_n) = (x_1, \dots, x_m, 0, \dots, 0)$, thus $\psi(f(M) \cap V) = \{x_{m+1} = \dots = x_n = 0\} \cap \varphi(V)$. $\square$

Remark. Conversely, for any submanifold $Z \subset N$, the inclusion $Z \subset N$ is an embedding.

5.3 Regular Value

Definition 5.3. Let $f : M \to N$ be a map of manifolds, $q \in N$ is called **regular value** if all points $p \in f^{-1}(q)$ satisfy that $D_p f$ are surjective.

Remark. By a theorem of Sard, the set of regular values of a map is dense (in N).

Fact (Sard's Theorem)**.** The set of regular values of a smooth map is dense in the target manifold.

Example 5.2. If $\dim M < \dim N$,

- every point not in the image of f is a regular value (this always holds);
- every point in the image of f is not a regular value.

Example 5.3. Let

$$\begin{array}{lll} f : \mathbb{R}^2 \to \mathbb{R} & \rightsquigarrow & D_{(x,y)}f : \mathbb{R}^2 \to \mathbb{R} \\ (x,y) \mapsto x \cdot y & & \quad \shortparallel \\ & & (b,a) \text{ has full rank iff } (a,b) \neq (0,0) \end{array}$$

Theorem 5.4. If $f : M \to N$ is smooth and $p \in N$ is regular value, then $f^{-1}(p)$ is a submanifold of M.

Proof. Let $q \in f^{-1}(p)$. Then by the local form for submersions, we find charts (U, φ), (V, ψ) around q, p, s.t. $\psi \circ f \circ \varphi^{-1}(x_1, \dots, x_m) = (x_1, \dots, x_n)$ is the projection. But then $\varphi(f^{-1}(p) \cap U) = \{x_1 = \cdots = x_n = 0\} \cap \varphi(U)$. □

5.4 Whitney's Embedding Theorem

Theorem 5.5 (Whitney's Embedding Theorem)**.** Every smooth manifold of dimension n can be embedded into $\mathbb{R}^{2n}$.

Remark.

- In general, this dimension is optimal, e.g. non-orientable surfaces ($\mathbb{RP}^2$, Klein bottle) cannot be embedded into $\mathbb{R}^3$ (but immersed). For particular manifold, better bonds on the dimension are possible, e.g. $S^1 \times S^1 \hookrightarrow \mathbb{R}^3$ or $\mathbb{R}^2 \overset{\mathrm{Id}}{\hookrightarrow} \mathbb{R}^2$.
- Any m-dimensional manifold can be immersed into $\mathbb{R}^{2m-a(m)}$, where $a(2m)$ is the number at 1's in the binary expansion of m.

We will only prove the following weaker version.

Theorem 5.6 (Weak Whitney's Theorem)**.** Every compact m-dimensional smooth manifold can be embedded into $\mathbb{R}^{2m+1}$.

Proof. Let X be a compact smooth m-dimensional manifold.

Claim 5.7. X can be embedded into some $\mathbb{R}^k$ for $k \gg 0$.

Proof. Let $\{(U_i, \varphi_i)\}_{i=1}^n$ be a finite atlas for X. Choose a partition of unity $\{\rho_i\}$ subordinate to $\{U_i\}_{i=1}^n$.

Next, define $\phi : X \to \mathbb{R}^k$ with $k = n(m+1)$

$$p \mapsto (\rho_1(p) \cdot \varphi_1(p), \ldots, \rho_n(p) \cdot \varphi_n(p), \rho_1(p), \ldots, \rho_n(p))$$

Then ϕ is an embedding. □

In fact, $\underline{\phi \text{ is injective}}$: Let $\phi(p_1) = \phi(p_2)$. Choose i, s.t.

$$\rho_i(p_1) = \rho_i(p_2) \neq 0$$

Then

$$\begin{aligned} \rho_i(p_1) \cdot \varphi_i(p_1) = \rho_i(p_2) \cdot \varphi_i(p_2) &\Rightarrow \varphi_i(p_1) = \varphi_i(p_2) \\ &\xRightarrow{\varphi_i \text{ different}} p_1 = p_2 \end{aligned}$$

$\underline{D_p\phi \text{ is injective at all } p \in M:}$

$D_p\phi : T_pX \to T_{\phi(p)}\mathbb{R}^k \cong \mathbb{R}^k$

$D_p\phi = (D_p\rho_1 \cdot \varphi_1(p) + \rho_1(p) \cdot D_p\varphi_1, \ldots, D_p\rho_n(p) + \rho_n(p) \cdot D_p\varphi_n, D_p\rho_1, \ldots, D_p\rho_n)$

$$\begin{aligned} \text{Thus if } (D_p\phi)(X) = 0 \text{ where } X \in T_pX \ &\Rightarrow (D_p\rho_i)(X) = 0,\ \forall i \\ &\Rightarrow \rho_i(p) D_p\varphi_i(X) = 0,\ \forall i \\ &\xRightarrow{\varphi_i \text{ different}} X = 0 \end{aligned}$$

So D_pf_i is injective.

Lemma 5.8. If $f : A \to B$ is an injective immersion of smooth manifold and A is compact, then f is an embedding.

Proof. We need to show f is a closed map.

$$\begin{aligned} \text{If } Z \subset A \text{ is closed } &\xRightarrow{A \text{ compact}} Z \text{ is compact} \\ &\xRightarrow{f \text{ continuous}} f(Z) \text{ compact} \\ &\xRightarrow{B \text{ Hausdorff}} f(Z) \text{ closed} \end{aligned}$$

□

Claim 5.9. If an m-manifold admits an injective immersion into $\mathbb{R}^k$ with $k > 2m+1$, then it admits an injective immersion into $\mathbb{R}^{k-1}$.

Proof. The idea is to project onto a generic hyperplane.

Hyperplanes are described via their normal vectors: For $[v] \in \mathbb{RP}^{k-1}$ denote by $P_{[v]} = \{u \in \mathbb{R}^k \mid \langle u, v \rangle = 0\}$ the hyperplane orthogonal to $[v]$ and by $\pi_{[v]} : \mathbb{R}^k \to P_{[v]}$ the orthogonal projection.

Write $\phi_{[v]} := \pi_{[v]} \circ \phi : X \to \mathbb{R}^{k-1}$.

Claim: For a generic choice of $[v]$, $\phi_{[v]}$ will be an injective immersion.

Assume $\phi_{[v]}$ is not injective, i.e. there are $p_1 \neq p_2 \in X$, s.t. $\phi_{[v]}(p_1) = \phi_{[v]}(p_2)$ and so $\phi(p_1) - \phi(p_2)$ lies in the line $[v]$, i.e. the points where $\phi_{[v]}$ is not injective live in the image of

$$\begin{aligned} (X \times X) \setminus \Delta_x &\to \mathbb{RP}^{k-1} \\ (p_1, p_2) &\mapsto [\phi(p_1) - \phi(p_2)] \end{aligned}$$

where $\Delta_x = \{(x,x)\}$.

By Sard's theorem, for a set containing an open dense set of $[v]$'s, $\phi_{[v]}$ will be injective.

Similarly, consider a $[V]$, s.t. there exists $p \in X$ with $D_p\phi_{[v]}$ not injective, i.e. there exists $0 \neq A \in T_pX$, s.t. $\underbrace{D_p\phi_{[v]}}_{D_p(\pi_{[v]}\circ\phi)}(A) = 0 \Leftrightarrow (\pi_{[v]} \circ D_p\phi)(A) \Leftrightarrow$ $(D_p\phi)(A)$ is contained in the line $[V]$.

Remark. $X \subset TX$ submanifold via $x \mapsto (x,0)$.

i.e. the $[v]$'s, s.t. $\phi_{[v]}$ is not an immersion live in the image of

$$\begin{aligned} TX \setminus X &\to \mathbb{RP}^{k-1} \\ (p,A) &\mapsto (D_p\phi)(A) \end{aligned}$$

where $p \in X$, $A \in T_pX$. Again by Sard's theorem, the set s.t. $\phi_{[v]}$ is an immersion, is open dense. ☐

Now take a $[V]$ in the intersection of these dense sets. ☐

Chapter 6

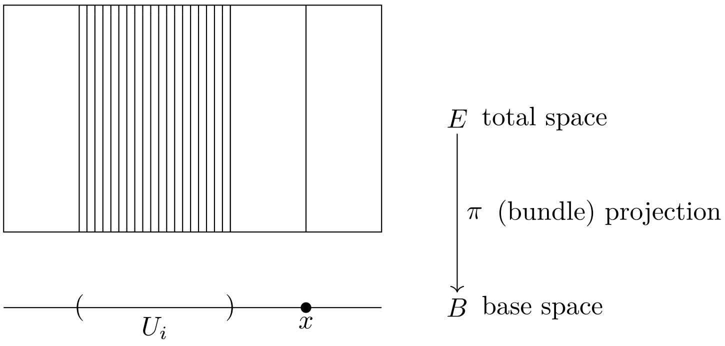
Smooth Vector Bundles

6.1 Vector Bundles

Definition 6.1. A smooth vector bundle of rank k is a pair of smooth manifolds E, B together with a submersion $\pi : E \to B$, s.t. the following hold:

(1) for every $x \in B$, the fibre $\pi^{-1}(x)$ has the structure of a k-dimensional $\mathbb{R}$-vector space.

(2) B has an open cover $\{U_i \mid i \in I\}$ and diffeomorphisms $\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^k$ which restrict to linear isomorphisms on every $\pi^{-1}(x)$, $x \in U_i$ and satisfy $\pi_1 \circ \psi_i = \pi$.



where $\pi^{-1}(x) = E_x$ is the fibre over $x \in B$.

$$\dim E = \dim B + \dim E_x = \dim B + k$$

$$\begin{tikzcd}
\pi^{-1}(U_i) \arrow[rr, "\psi_i"] \arrow[dr, "\pi"'] & & U_i \times \mathbb{R}^k \arrow[dl, "\pi_1"] \\
& U_i &
\end{tikzcd}$$

$U_i \cap U_j = \varnothing$.

$$\begin{tikzcd}
(U_i \cap U_j) \times \mathbb{R}^k \arrow[rr, bend right, "\psi_j \circ \psi_i^{-1}"'] & \pi^{-1}(U_i \cap U_j) \arrow[l, "\psi_i"'] \arrow[r, "\psi_j"] & (U_i \cap U_j) \times \mathbb{R}^k \\
(x, v) \arrow[rr, mapsto] & & (x, \underbrace{\gamma_{ji}(x)}_{\in GL_k(\mathbb{R})} (x)(v))
\end{tikzcd}$$

$U_i \cap U_j \cap U_k \neq \varnothing \Rightarrow \gamma_{ji} \circ \gamma_{il} = \gamma_{jl}$. Setting $j = l$ gives $\gamma_{ji} = \gamma_{ij}^{-1}$. $\gamma_{ii} = \mathrm{Id}$, $\forall i \in I$. From the open covering of B by the U_i and the transition maps γ_{ij}, one can reconstruct the vector bundle $\pi : E \to B$.

Definition 6.2. Let $\pi : E \to B$, $\pi' : E' \to B$ be smooth vector bundles over the same base B. An **isomorphism** of vector bundles is a diffeomorphism $f : E \to E'$ which is a linear isomorphism on every fibre and satisfy $\pi' \circ f = \pi$, i.e.

$$\begin{tikzcd}
E \arrow[rr, "f"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\
& B &
\end{tikzcd}$$

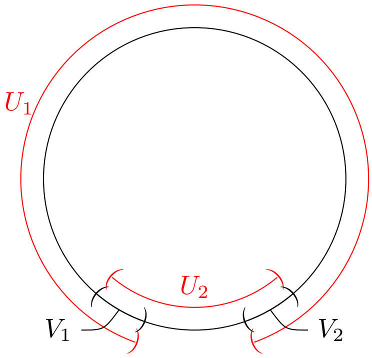
Example 6.1.

(1) Product bundles $E = B \times \mathbb{R}^k$, $\pi = \pi_1$.

Definition 6.3. A vector bundle is **trivial** if it is isomorphic to a product bundle.

(2) Let $B = M$ be any smooth manifold, $E = TM$ is a vector bundle of rank$=\dim M$.

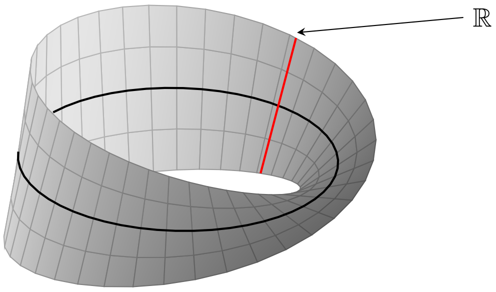
(3) Let $B = S^1$ and take $U_1 \times \mathbb{R}$, $U_2 \times \mathbb{R}$. Then $U_1 \cap U_2 = V_1 \sqcup V_2$.



$$\begin{aligned}
\gamma_{ij} :& U_i \cap U_j \to GL_k(\mathbb{R}) \subset \mathbb{R}^{k^2} \text{ smooth} \\
\gamma_{12} :& U_1 \cap U_2 = V_1 \sqcup V_2 \to GL_1(\mathbb{R}) = \mathbb{R}^* \text{ ($\mathbb{R}$ without origin)}
\end{aligned}$$

$$x \mapsto \begin{cases} 1 & \text{for } x \in V_1 \\ -1 & \text{for } x \in V_2 \end{cases}$$

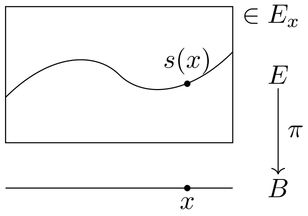
Construct E from this structure cocycle. Then E is the **Möbius strip**.



Rank 1 vector bundles over S^1: $S^1 \times \mathbb{R}$, TS^1, $E = M$. Then $S^1 \times \mathbb{R}$ is isomorphic to TS^1, but TS^1 is not isomorphic to $E = M$.

Definition 6.4. Let $\pi : E \to B$ be a vector bundle. A **section** of E is a smooth map $s : B \to E$, s.t. $\pi \circ s = \mathrm{Id}_B$.

$$\begin{tikzcd} B \arrow[d, "s"'] \arrow[dr, "\mathrm{Id}"] & \\ E \arrow[r, "\pi"'] & B \end{tikzcd}$$



Lemma 6.1. A vector bundle $E \xrightarrow{\pi} B$ of rank k is trivial if and only if it admits k sections $s_1, \dots, s_k \in \Gamma(E)$ which are pointwise linearly independent, where $\Gamma(E) = \{s : B \to E \mid \pi \circ s = \mathrm{Id}_B\}$ is a $\mathbb{R}$-vector space and a $\mathcal{C}^\infty(B)$-module.

Proof. First, assume E is trivial, and $f : E \to B \times \mathbb{R}^k$ is an isomorphism. Define $s_i(x) := f^{-1}(x, e_i) \in \Gamma(E)$, where $e_1, \dots, e_k$ is any basis of $\mathbb{R}^k$. Then $s_1, \dots, s_k$ are pointwise linearly independent.

Second, suppose $s_1, \dots, s_k$ are linearly independent sections. Define

$$\begin{aligned} g : B \times \mathbb{R}^k &\to E_k \\ (x, (\lambda_1, \dots, \lambda_k)) &\mapsto \sum_{i=1}^{k} \lambda_i s_i(x) \end{aligned}$$

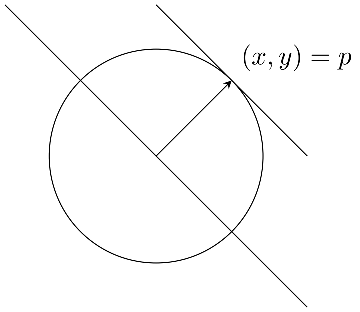
This is a smooth map and satisfies $\pi \circ g \to \mathrm{Id}_B$.

Moreover, g is a linear isomorphism $x \times \mathbb{R}^k \to E_x$, $\forall x \in B$. $f := g^{-1}$ is a global trivialization of E. ☐

Corollary 6.2. A rank 1 vector bundle is trivial if and only if it has a nowhere zero section, i.e. $\exists s \in \Gamma(E)$, s.t. $s(x) \neq 0$, $\forall x \in B$.

Remark. The zero $0 \in \Gamma(E)$ is the section $\begin{array}{l} 0 : B \to E \\ x \mapsto 0 \in E_x \end{array}$. This is called the **zero-section**.

Let $S^1 \subset \mathbb{R}^2$ be the unit circle as the following figure shown.



Then $TS^1 \subset T\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}^2$ and we have

$$\begin{aligned} T_pS^1 &= \mathbb{R} \cdot (-y, x) \\ TS^1 &= \{(x, y, s, t) \in \mathbb{R}^4 \mid x^2 + y^2 = 1, s = -\lambda y, t = \lambda x, \text{for some } \lambda \in \mathbb{R}\} \end{aligned}$$

with the map

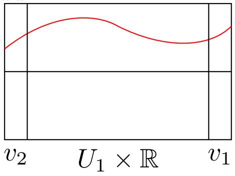
$$\begin{tikzcd} TS^1 \arrow[r] & S^1 \arrow[l, bend right, "s"'] \end{tikzcd}$$

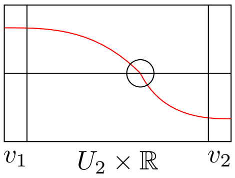
$$(x, y, s, t) \longmapsto (x, y)$$

where $s(x, y) = (x, y, -y, x)$.

Lemma 6.3. The Möbius strip M is **not** a trivial vector bundle.

Proof. Suppose M were trivial. Then let $s : S^1 \to M$ be a nowhere zero-section.





s is smooth hence it is continuous. The intermediate value theorem says it has a zero. This leads to a contradiction. ☐

6.2 Metric

Definition 6.5. A **metric** on a vector bundle $\pi : E \to B$ is a fibrewise positive definite scalar product on E_x which depends smoothly on $x \in B$.

Smoothness can be checked/defined in one of two ways:

(1) With local trivialization:

Let $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ be a local trivialization with $x \in U$. A metric

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r] & U \times \mathbb{R}^k \\ E_y \arrow[u, phantom, "\cup"] \arrow[r, "\cong"] & \{y\} \times \mathbb{R}^k \end{tikzcd}$$

$\langle\ ,\ \rangle$ on E induces a scalar product $\langle\ ,\ \rangle_x$ on E_x, which we think of as a scalar product g_x on $\mathbb{R}^k$, via the isomorphism $E_x \underset{\psi}{\cong} \mathbb{R}^k$. g_y, as y varies in

U, gives a family of positive definite scalar products on $\mathbb{R}^k$, dependency on y.

$$\begin{aligned} g : U &\to V = \text{symmetric bilinear forms on } \mathbb{R}^k. \\ y &\mapsto g_y \end{aligned}$$

Smoothness of $\langle\ ,\ \rangle$ means that in every local trivialization, g is a smooth map.

(2) Smoothness of $\langle\ ,\ \rangle$ means that for any two $s_1, s_2 \in \Gamma(E)$, $\langle s_1, s_2\rangle \in \mathcal{C}^\infty(B)$.

$$\begin{aligned} \langle s_1, s_2\rangle : B &\to \mathbb{R} \\ x &\mapsto \langle s_1(x), s_2(x)\rangle_x \end{aligned}$$

Proposition 6.4. Every vector bundle admits a metric.

Proof. Let $\{U_i \mid i \in I\}$ be a covering of B by trivializing open sets for E, $\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^k$.

For $y \in U_i$, let $\langle\ ,\ \rangle_{i,y}$ be the scalar product on E_y obtained from the standard scalar product on $\mathbb{R}^k$ via the isomorphism $\psi : E_y \to \{y\} \times \mathbb{R}^k$.

Let ρ_i be a partition of unity subordinate to the covering of B by the U_i. Define $\langle\ ,\ \rangle := \sum\limits_i \rho_i \cdot \langle\ ,\ \rangle_i$. This is a metric! It satisfies $\sum \rho_i \equiv 1$. ☐

Remark. This proof uses positive-definiteness.

6.3 Constructions with Vector Bundles

(1) Subbundles

If $\pi : E \to B$ is a vector bundle of rank k, then a subbundle of rank $l \leqslant k$ is a submanifold $F \subset E$ such that $\pi\big|_F : F \to B$ is a vector bundle of rank l. For every $x \in B$, $F \cap E_x = F_x$ is a l-dimensional subspace of $E_x \cong \mathbb{R}^k$.

Let $\pi : E \to B$, $\pi' : E' \to B$ be vector bundles and $f : E \to E'$ a smooth map with $\pi' \circ f = \pi$ and $f\big|_{E_x}$ is linear for all $x \in B$.

If $\operatorname{rank}(f\big|_{E_x})$ is a constant function of $x \in B$, then $\operatorname{im}(f) \subset E'$ is a subbundle of rank $= \operatorname{rank}(f)$ and $\ker(f) \subset E$ is a subbundle of rank $= \operatorname{rank} E - \operatorname{rank} f$.

(2) Quotient bundles

If E is a vector bundle, $F \subset E$ a subbundle, the $\bigcup\limits_{x \in B} (E_x/F_x)$ is a vector bundle over B, called the **quotient bundle**.

(3) If E has a metric $\langle\ ,\ \rangle$, $F^\perp = \{v \in E_x \mid x \in B, \langle v, w\rangle_x = 0, \forall w \in F_x\}$ is a subbundle, and $F^\perp \cong E/F$.

(4) Whitney sums

$E \xrightarrow{\pi} B$, $E' \xrightarrow{\pi'} B$ are vector bundles.

$E \oplus E' \to B$ is the vector bundle with $(E \oplus E')_x = E_x \oplus E'_x$, for all $x \in B$.

Let $\{U_i \mid i \in I\}$ be an open cover of B which is simultaneously trivialization for E and for E'. Let $\gamma_{ij} : U_i \cap U_j \to GL_k(\mathbb{R})$, $\gamma'_{ij} : U_i \cap U_j \to GL_{k'}(\mathbb{R})$

be the corresponding cocycles of transition maps. Then $E \oplus E'$ is the vector bundle of rank $k + k'$ defined by

$$\begin{aligned} U_i \cap U_j &\to GL_{k+k'}(\mathbb{R}) \\ x &\mapsto \begin{pmatrix} \gamma_{ij}(x) & 0 \\ 0 & \gamma'_{ij}(x) \end{pmatrix} \end{aligned}$$

(5) Dual bundles

If $\pi : E \to B$ is a vector bundle of rank k, the dual bundle $E^* \xrightarrow{\pi} B$ is the rank k vector bundle given by $\gamma_{ij}(x) \in GL_k(\mathbb{R}) = \mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^k)$.

$$\begin{aligned} \lambda &: \mathbb{R} \to \mathbb{R}^k \\ \lambda^* &: (\mathbb{R}^k)^* \mapsto (\mathbb{R}^k)^* \text{ defined by } \lambda^*(\varphi)(x) = \varphi(\lambda(x)) \\ &\mathrm{Hom}((\mathbb{R}^k)^*, (\mathbb{R}^k)^*) \ni GL_k(\mathbb{R}) \ni \gamma^*_{ij}(x) \end{aligned}$$

If $F \subset E$ is a subbundle, then

$$\begin{tikzcd} F \oplus F^{\perp} \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\wr\|"] & E \\ F \oplus (E/F) \end{tikzcd}$$

Let $G \subset GL_k(\mathbb{R})$ be a subgroup.

Definition 6.6. A **G-structure** on a rank k vector bundle $E \xrightarrow{\pi} B$ is a system of local trivializations whose transition maps take values in G.

Remark. A G-structure is sometimes called a **G-reduction**.

(1) $G = \{e\}$.

In this case, a G-structure is a global trivialization.

(2) $G = GL_k^+(\mathbb{R})$ orientation-preserving isomorphism $\mathbb{R}^k \xrightarrow{\cong} \mathbb{R}^k$. In this case, a G-structure on E is an orientation for E, i.e. a consistent choice of orientation for all E_x varying smoothly $x \in B$.

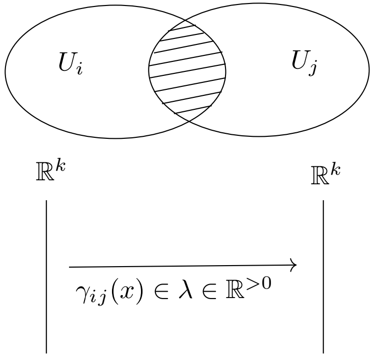
The Möbius strip as a vector bundle over S^1 does not admit an orientation.

Lemma 6.5. A rank 1 vector bundle is trivial if and only if it is orientable.

Proof. If E is trivial, then it is orientable. Conversely, suppose E is orientable and of rank 1. Then E has a G-structure for $GL_1^+(\mathbb{R}) = \mathbb{R}^{>0}$.

$$\gamma_{ij} : U_i \cap U_j \to \mathbb{R}^{>0}$$

Without loss of generality, all $U_i \cap U_j$ are either $\varnothing$ or diffeomorphic to bundles. Then we can define the γ_{ij} smoothly to be $\equiv 1$. Then E is trivial since it has a G-structure for the trivial group.



□

(3) $G = O(k)$.

In this case, a G-structure is a choice of metric $\langle\ ,\ \rangle$ on E. Every E admits such a G-structure.

(4) $G = SO(k) = GL_k^+(\mathbb{R}) \cap O(k)$.

6.4 Pullback Bundles

Suppose $f : M \to N$ is a smooth map, and $\pi : E \to N$ is a smooth vector bundle over N.

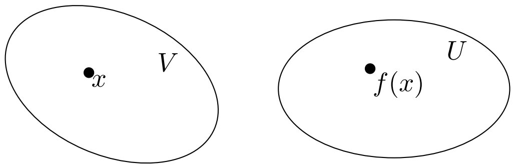
Definition 6.7. $f^*E := \{(x, v) \in M \times E \mid f(x) = \pi(v)\}$ is the pullback bundle of E under f.

That is the following diagram commute:

$$\begin{tikzcd}
f^*E \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & E \arrow[d, "\pi"] \\
M \arrow[r, "f"'] & N
\end{tikzcd}$$

And we have

$$\pi_1^{-1}(x) = \pi^{-1}(f(x)) = E_{f(x)}$$



$$\pi^{-1}(U) \cong U \times \mathbb{R}^k$$

If E is a vector bundle of rank k over N, then f^*E is a vector bundle of rank k over M.

6.5 Bundles Homomorphisms

Definition 6.8. If $\pi_E : E \to M$, $\pi_F : F \to N$ are smooth vector bundles, then a homomorphism of vector bundles is a smooth map $h : E \to F$, which restricts to every $E_x \subset E$ as a linear map into a fibre of F.

$f(x) = \pi_F \circ h(v)$ for any $v \in E_x$. This is well-defined and smooth.

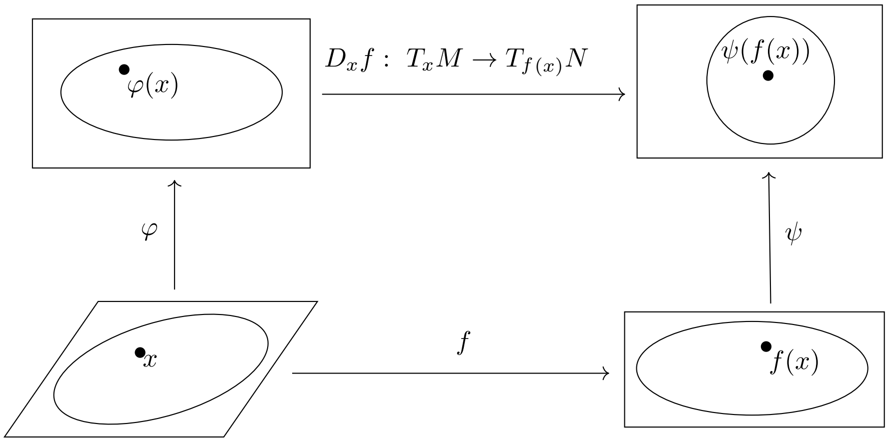
$$\begin{tikzcd}
E \arrow[r, "h"] \arrow[d, "\pi_E"'] & F \arrow[d, "\pi_F"] \\
M \arrow[r, "f"'] & N
\end{tikzcd}$$

commute

Example 6.2. $\pi_2 : f^*E \to E$ is a homomorphism of vector bundle.

Example 6.3. If $f : M \to N$ is any smooth map, then $Df : TM \to TN$ is a homomorphism of vector bundle.

$$\begin{tikzcd}
TM \arrow[d] & \\
M \arrow[r, "f"'] & N
\end{tikzcd}$$



Let $h : E \to F$ be a homomorphism of vector bundles covering $f : M \to N$.

$$\begin{tikzcd}
E \arrow[r, "\overline{h}"'] \arrow[rr, bend left, "h"] \arrow[dr, "\pi_E"'] & f^*F \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] & F \arrow[d, "\pi_F"] \\
& M \arrow[r] & N
\end{tikzcd}$$

Define $\overline{h} : E \to f^*F$ by $\overline{h}(v) = (\pi_E(v), h(v)) \in M \times F$. Then

$$\pi_1(\overline{h}(v)) = \pi_1(\pi_E(v), h(v)) = \pi_E(v)$$

$$\begin{tikzcd}
TM \arrow[r, "\overline{Df}"'] \arrow[rr, bend left, "Df"] \arrow[dr] & f^*TN \arrow[r] \arrow[d] & TN \arrow[d] \\
& M \arrow[r, "f"] & N
\end{tikzcd}$$

Let $N = \mathbb{R}$, then

$$\begin{tikzcd}
& M \times \mathbb{R} \arrow[d, phantom, "\cong" rotate=90] & \\
TM \arrow[ur, "Df"] \arrow[r] \arrow[dr] & f^*T\mathbb{R} \arrow[r] \arrow[d] & T\mathbb{R} \arrow[d] \\
& M \arrow[r, "f"] & \mathbb{R}
\end{tikzcd}$$

where $Df : TM \to M \times \mathbb{R}$ and $T\mathbb{R} = \mathbb{R} \times \mathbb{R}$, the first $\mathbb{R}$ represents

$$v \mapsto (\pi(v),\ \underbrace{(D_{\pi(v)}f)(v)}_{\substack{\text{linear form in} \\ \text{tangent space}}})$$

manifold and the second $\mathbb{R}$ represents vector space.

If $f : M \to \mathbb{R}$ is smooth, then its derivative Df is a section in $\mathrm{Hom}(TM, M \times \mathbb{R}) = T^*M = (TM)^*$. Three different interpretation of derivative of smooth function:

$$Df : TM \to T\mathbb{R} \qquad\qquad Df : TM \to M \times \mathbb{R} \qquad\qquad \mathrm{d}f \in \Gamma(T^*M)$$

Chapter 7

Flows

7.1 Velocity Vector

Let M be a smooth manifold, $c : \mathbb{R} \to M$ a smooth map. (c is called smooth curve.)

Definition 7.1. $\dot{c}(t) \in T_{c(t)}M$ is defined by

$$D_t c(1) = D_t c\left(\frac{\partial}{\partial t}\right)$$

where $T\mathbb{R} = \underset{\left(t, \lambda \cdot \frac{\partial}{\partial t}\right)}{\qquad} \mathbb{R} \times \mathbb{R}$. This is the **velocity vector** of c at t (at $c(t)$).

Example 7.1. $M = \mathbb{R}^n$, then $c(t) = (x_1(t), \dots, x_n(t))$.

$$\dot{c}(t) = (\dot{x}_1, \dots, \dot{x}_n) = \left(\frac{\partial x_1}{\partial t}, \dots, \frac{\partial x_n}{\partial t}\right) \in T_{c(t)}\mathbb{R}^n = \mathbb{R}^n$$

7.2 Global Flows

Definition 7.2. A (global) flow on a smooth manifold M is a smooth map

$$\varphi : M \times \mathbb{R} \to M$$

satisfying the following properties:

$$\left.\begin{aligned} \varphi(x, 0) &= x \\ \varphi(\varphi(x, t), s) &= \varphi(x, t + s) \end{aligned}\right\} \forall x \in M,\ t, s \in \mathbb{R}$$

Write $\varphi(x, t) = \varphi_t(x)$, then

$$\left.\begin{aligned} \varphi_0 &= \mathrm{Id}_M \\ \varphi_t \circ \varphi_s &= \varphi_{t+s} \end{aligned}\right\} \Rightarrow \varphi_{-t} = (\varphi_t)^{-1}$$

Every φ_t is a smooth map $M \to M$ with a smooth inverse, so $\varphi_t \in \mathrm{Diff}(M)$. A flow φ defines a group homomorphism: $\mathbb{R} \to \mathrm{Diff}(M)$.

Definition 7.3. $\mathfrak{X}(M) := \Gamma(TM)$ is the vector space of **vector fields** on M.

Given a flow φ, we can define $X \in \mathfrak{X}(M)$ by

$$X_p = \left(\frac{\partial}{\partial t} \varphi_t(p) \right) \bigg|_{t=0} = (D_0 \varphi) \left(\frac{\partial}{\partial t} \right)$$

where

$$\begin{aligned} \varphi : \mathbb{R} &\to M \\ t &\mapsto \varphi_t(p) \end{aligned}$$

If $p = \varphi_s(q)$, then

$$\begin{aligned} X_p &= \left(\frac{\partial}{\partial t} \varphi_t(p) \right) \bigg|_{t=0} = \left(\frac{\partial}{\partial t} \varphi_{t+s}(q) \right) \bigg|_{t=0} \\ &= \left(\frac{\partial}{\partial t} \varphi_s(\varphi_t(q)) \right) \bigg|_{t=0} = D_q \varphi_s(X_q) \end{aligned}$$

Lemma 7.1. *The vector field $X \in \mathfrak{X}(M)$ obtained by differentiating a flow φ is invariant under $D\varphi$.*

7.3 Local Flows

Let M be a smooth manifold.

Definition 7.4. A **local flow** on M is a covering of M by open sets U_i and a family of smooth maps

$$\varphi^i : U_i \times (-\varepsilon_i, \varepsilon_i) \to M$$

s.t. $\varphi^i_0 = \mathrm{Id}_{U_i}$ and $\varphi^i_t \circ \varphi^i_s = \varphi^i_{t+s}$ whenever all 3 terms are defined.

Proposition 7.2. *For every vector field $X \in \mathfrak{X}(M)$, there exists a local flow $\{U_i \mid i \in I\}$, φ_i such that*

$$\frac{\partial}{\partial t} \varphi^i_t(p) \bigg|_{t=0} = X_p$$

whenever $p \in U_i$.

Proof. The statement is local in M, so we can work in a chart, so locally in $\mathbb{R}^n$. Using coordinates in $\mathbb{R}^n$, we need to solve locally a linear system of ODEs with $\mathcal{C}^\infty$ coefficients. This can be done! □

If $U_i \cap U_j \neq \varnothing$, then we require $\varphi^i_t(x) = \varphi^j_t(x)$ for all $x \in U_i \cap U_j$ and $|t| < \min\{\varepsilon_i, \varepsilon_j\}$.

Given $X \in \mathfrak{X}(M)$, we can locally integrate X to get a local flow in this sense.

Definition 7.5. Two local flows are equivalent if their union is also a local flow.

This is an equivalent relation!

Proposition 7.3. *There is a one-to-one corresponding between equivalence classes of local flows on M and vector fields $X \in \mathfrak{X}(M)$.*

(U_i, φ^i), $i \in I \rightsquigarrow X \rightsquigarrow (V_j, \varphi^j)$, $j \in J$ equivalent to (U_i, φ^i), $i \in I$.
$X \rightsquigarrow (V_j, \varphi^j)$, $j \in J \rightsquigarrow X$.

Definition 7.6. A vector field X is **complete** if there is a local flow $\varphi^i : U_i \times \mathbb{R} \to M$ in the corresponding equivalence class.

Under the one-to-one correspondence in the Proposition 7.3, complete vector fields give global flows $\varphi : M \times \mathbb{R} \to M$, where $\varphi(x,t) := \varphi^i(x,t)$ if $x \in U_i$.

Proposition 7.4. If $X \in \mathfrak{X}(M)$ has compact support

$$\operatorname{supp}(X) := \overline{\{x \in M \mid X(x) \neq 0\}}$$

then it is complete.

Proof. Step 1: Consider a local flow (U_i, φ^i) for X, $i \in I$. Since the U_i cover M, they cover $\operatorname{supp}(X)$. Since $\operatorname{supp}(X)$ is compact, there exist finitely many U_i, say $U_1, \ldots, U_k$, such that $\operatorname{supp}(X) \subset \bigcup\limits_{i=1}^{k} U_i$.

Let $U_0 := M \setminus \operatorname{supp}(X) \underset{\text{open}}{\subset} M$.

Define $\varphi^0 : U_0 \times \mathbb{R} \to M$, $\forall x \in U_0, t \in \mathbb{R}$.
$(x,t) \mapsto x$

$U_0, U_1, \ldots, U_k$ form a covering of M, and the pair (U_i, φ^i), $i \in \{0, \ldots, k\}$ are a local flow for X. Set $\varepsilon := \min\{\varepsilon_1, \ldots, \varepsilon_k\} > 0$.

$\varphi_t(x) = \varphi^i_t(x)$ is defined for all $x \in M$ and all $|t| < \varepsilon$.

Step 2: Let X be any vector field which admits a local flow (U_i, φ^i) defined for all times $|t| < \varepsilon$. Then we can define $\varphi^i_{Nt}(x) := \underbrace{\varphi^i_t \circ \cdots \circ \varphi^i_t(x)}_{N \text{ times for all } N \in \mathbb{N},\ |t| < \varepsilon}$.

$\varphi_{s+t} = \varphi_s \circ \varphi_t$ whenever both are defined. ☐

Corollary 7.5. If M is compact, then all $X \in \mathfrak{X}(M)$ are complete.

Example 7.2. Compact support is sufficient for completeness, but not necessary.
$\dfrac{\partial}{\partial x_1} \neq 0$ everywhere

Example 7.3. $M = \mathbb{R}^n \setminus \{0\}$, $X = \dfrac{\partial}{\partial x_1} \neq 0$ everywhere
If $p = (-s, 0, \ldots, 0)$, $s > 0$, then $\varphi(p,t)$
is not defined for $t \geqslant s$.

Chapter 8

Lie Theory

8.1 Lie Derivative

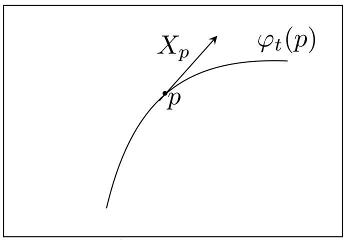
Let M be a smooth manifold, $f \in \mathcal{C}^\infty(M) = \mathcal{C}^\infty(M, \mathbb{R})$ and $X \in \mathfrak{X}(M)$.

Definition 8.1. $(L_X f)(p) = \frac{\partial}{\partial t} \varphi_t^*(f)(p)\Big|_{t=0}$, where φ is the flow of X.

$$= \frac{\partial}{\partial t} f(\varphi_t(p))\Big|_{t=0} = \lim_{t \to 0} \frac{f(\varphi_t(p)) - f(\varphi_0(p))}{t}$$

$$= \lim_{t \to 0} \frac{f(\varphi_t(p)) - f(p)}{t} = D_p f(X(p)) = d_p f(X(p))$$

$$\begin{array}{c} D_p f : T_p \to T_{f(p)}\mathbb{R} = \mathbb{R} \\ \updownarrow \\ d_p f : T_p^* M \end{array}$$



The Lie derivative L_X sends smooth functions to smooth functions

$$L_X : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$$

Lemma 8.1. $L_X(f \cdot g) = (L_X f) \cdot g + f \cdot (L_X g)$ for all $f, g \in \mathcal{C}^\infty$, where

$$f \cdot g : M \to \mathbb{R}$$

Like Leibniz rule in derivative $(fg)' = f'g + fg'$, we have

$$D_p(f \cdot g) = (D_p f) \cdot g + f \cdot (D_p g)$$

We can see that

$$\begin{aligned} L : \mathfrak{X}(M) &\to \mathrm{Der}(\mathcal{C}^\infty(M)) \\ X &\mapsto L_X \end{aligned}$$

Definition 8.2. If A is a $\mathbb{R}$-algebra, then

$$\mathrm{Der}(A) := \{d : A \to A \mid d \text{ is } \mathbb{R}\text{-linear and } d(a \cdot b) = d(a) \cdot b + a \cdot d(b)\}$$

If $\lambda \in \mathbb{R}$, then $L_{\lambda X} f = \lambda L_X f$, $\forall f \in \mathcal{C}^\infty(M)$. In fact, for all $g \in \mathcal{C}^\infty(M)$, $L_{gX} f = g L_X f$, $\forall f \in \mathcal{C}^\infty(M)$. Moreover, $L_{X+Y}(f) = L_X f + L_Y f$.

Proof. $\forall p \in M$, we have

$$\begin{aligned} L_X(f \cdot g)(p) &= \frac{\partial}{\partial t}(f \cdot g)(\varphi_t(p))\bigg|_{t=0} \\ &= \frac{\partial}{\partial t}(f(\varphi_t(p)) \cdot g(\varphi_t(p)))\bigg|_{t=0} \\ &= (L_X f)(p) \cdot g(p) + f(p) \cdot (L_X g)(p) \end{aligned}$$

☐

Proposition 8.2. The map $\begin{aligned} \mathfrak{X}(M) &\to \mathrm{Der}(\mathcal{C}^\infty(M)) \\ X &\mapsto L_X \end{aligned}$ is an isomorphism of vector spaces.

Proof. (1) The map is linear.
(2) The map is injective: If $X \not\equiv 0$, then $\exists p \in M$, s.t. $X(p) \neq 0$. Consider $\varphi : U_p \times (-\varepsilon, \varepsilon) \to M$ be part of a local flow of x. $\exists f \in \mathcal{C}^\infty(U_p)$, s.t. $f(\varphi_t(p)) \equiv t$. After multiplication with a suitable bump function and extension by 0, we may arrange $f \in \mathcal{C}^\infty(M)$. $(L_X f)(p) = \left(\frac{\partial}{\partial t} t\right)\bigg|_{t=0} = 1$, so $L_X \not\equiv 0$.
(3) The map is surjective: Let $\Delta \in \mathrm{Der}(\mathcal{C}^\infty(M))$.
Step 1: If $U \subset M$ is open, and $f \in \mathcal{C}^\infty(M)$ is such that $f\Big|_U \equiv 0$, then $\Delta(f)\Big|_U \equiv 0$. For $x \in U$, take $\varphi \in \mathcal{C}^\infty(M)$ with $\varphi(x) = 0$ and $\varphi\Big|_{M \setminus U} \equiv 1$.

$$\begin{aligned} \Rightarrow \varphi \cdot f = f &\Rightarrow \Delta(f) = \Delta(\varphi) \cdot f + \Delta(f) \cdot \varphi \\ &\Rightarrow (\Delta f)(x) = (\Delta\varphi)(x) \cdot \underbrace{f(x)}_{\overset{\shortparallel}{0}} + (\Delta f)(x) \cdot \underbrace{\varphi(x)}_{\overset{\shortparallel}{0}} = 0 \end{aligned}$$

Step 2: If there is an open neighborhood U of a point $x \in M$, such that $f\Big|_U \equiv g\Big|_U$, then $(\Delta f)(x) = (\Delta g)(x)$. (Apply Step 1 to $f - g$.)
Step 3: Let G_x be the $\mathbb{R}$-vector space of germs of $\mathcal{C}^\infty$ functions at $x \in M$. We can define

$$\begin{aligned} \Delta(x) : G_x &\to \mathbb{R} \\ [f] &\mapsto (\Delta f)(x) \end{aligned}$$

Step 2 says that this is well-defined. $\Delta(x)$ is a derivation on the algebra G_x. Using a chart, we may assume $M = \mathbb{R}^n$, $x \in \mathbb{R}^n$, $\Delta(x) = \sum_{i=1}^n \lambda_i \frac{\partial}{\partial x_i}$. So $\Delta(x)$ is a tangent vector in $T_x M$, and it depends smoothly on x. Define $X \in \mathfrak{X}(M)$ by setting $X(x) = \Delta(x)$.

Thus, $\Delta = L_X$. ☐

Lemma 8.3. For X, $Y \in \mathfrak{X}(M)$, there is a unique $[X, Y] \in \mathfrak{X}(M)$, s.t. $L_X L_Y - L_Y L_X = L_{[X,Y]}$.

Proof.

$$\begin{aligned}(L_XL_Y - L_YL_X)(f\cdot g) &= L_X((L_Yf)\cdot g + f\cdot(L_Yg)) - L_Y((L_Xf)\cdot g + f\cdot(L_Xg))\\ &= (L_XL_Yf)\cdot g + \cancel{(L_Yf)\cdot(L_Xg)} + \cancel{(L_Xf)\cdot(L_Yg)} + f\cdot(L_XL_Yg)\\ &- (L_YL_Xf)\cdot g - \cancel{(L_Xf)\cdot(L_Yg)} - \cancel{(L_Yf)\cdot(L_Xg)} - f\cdot(L_YL_Xg)\\ &= (L_XL_Y - L_YL_X)(f)\cdot g + f\cdot(L_XL_Y - L_YL_X)(g)\end{aligned}$$

$\forall f, g \in \mathcal{C}^\infty(M)$, so $L_XL_Y - L_YL_X$ is a derivation on $\mathcal{C}^\infty(M)$. By the surjectivity in the Proposition 8.2, $\exists [X,Y] \in \mathfrak{X}(M)$, s.t.

$$L_{[X,Y]} = L_XL_Y - L_YL_X$$

By the injectivity in the Proposition 8.2, this vector field is unique. □

Definition 8.3. $[X,Y]$ is the **Lie bracket** of X and Y.
$[\ ,\ ] : \mathfrak{X}(M)\times\mathfrak{X}(M) \to \mathfrak{X}(M)$ is bilinear and skew-symmetric.

Lemma 8.4 (Jacobi Identity)**.** $[[X,Y],Z]+[[Z,X],Y]+[[Y,Z],X] = 0, \forall X, Y, Z \in \mathfrak{X}(M)$.

8.2 Lie Algebra and Lie Group

Definition 8.4. A **Lie algebra** $\mathfrak{g}$ is a $\mathbb{R}$-vector space, with a map $[\ ,\ ] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$, which is bilinear, skew-symmetric, and satisfies the Jacobi identity.

$\mathfrak{X}(M)$ is a Lie algebra with the Lie bracket.

Definition 8.5. A Lie group G is a smooth manifold with a group structure

$$\begin{aligned} m : G\times G &\to G\\ (g_1, g_2) &\mapsto g_1g_2 = g_1\cdot g_2\end{aligned}$$

s.t. m and $\begin{aligned} i : G &\to G \\ g &\mapsto g^{-1}\end{aligned}$ are smooth maps.

Example 8.1.

(1) $G = GL_k(\mathbb{R}) \subset \text{Mat}(k\times k, \mathbb{R}) = \mathbb{R}^{k^2}$.

(2) Subgroups of $GL_k(\mathbb{R})$ which are also submanifolds, e.g. $GL_k^+(\mathbb{R})$, $O(k)$, $GL_k(\mathbb{C}) \subset GL_{2k}(\mathbb{R})$.

If G is a Lie group and $g \in G$, then

$$\begin{aligned}\text{left multiplication } l_g : G &\to G\\ h &\mapsto g\cdot h\\ \text{right multiplication } r_g : G &\to G\\ h &\mapsto h\cdot g\end{aligned}$$

are diffeomorphisms.

$$\begin{tikzcd}
G \arrow[r] & G \times G \arrow[r, "m"] & G \\
h \arrow[r, mapsto] \arrow[rr, bend right, "l_g"'] & (g,h) \arrow[r, mapsto] & g \cdot h = l_g(h)
\end{tikzcd}$$

$l_{g^{-1}}$ is also a smooth map $l_g \circ l_{g^{-1}} = l_{g^{-1}} \circ l_g = \mathrm{Id}_G$.

For every $g \in G$,

$$D_e l_g : T_e G \to T_g G$$

is an isomorphism.
$\dim G = n$, $T_e G = \mathbb{R}^n$

$$\begin{aligned} G \times T_e G &\xrightarrow{t} TG \\ (g, v) &\mapsto (D_e l_g)(v) \end{aligned}$$

Lemma 8.5. This is an isomorphism of vector bundle, so TG is trivial.

Proof.

$$\begin{tikzcd}
G \times T_e G \arrow[r, "t"] \arrow[d, "\pi_1"'] & TG \arrow[d, "\pi"] \\
G \arrow[r, equal] & G
\end{tikzcd}$$

t is smooth. $D_e l_g$ is an isomorphism $T_e G \to T_g G$ for any $g \in G$. ☐

Definition 8.6. $X \in \mathfrak{X}(G)$ is **left-invariant** if $X(g) = (D_e l_g) X(e)$.

Lemma 8.6. If X is left-invariant, then $(D_g l_h(X(g)) = X(h \cdot g)$.

Proof. $((D_g l_h)(D_e l_g)(X(e))) = (D_e l_{h \cdot g})(X(e)) = X(h \cdot g)$. ☐

Definition 8.7. $\mathfrak{g} \subset \mathfrak{X}(G)$ is linear subspace of left-invariant vector field.

[,] sends pairs of left-invariant vector fields to a left-invariant vector field. $\Rightarrow \mathfrak{g} \subset \mathfrak{X}(G)$ is a sub-Lie algebra.

Definition 8.8. $\mathfrak{g} = L(G)$ is the **Lie algebra** of the Lie group G. $\dim \mathfrak{g} = \dim G$.

Definition 8.9. $X, Y \in \mathfrak{X}(M)$. φ_t is the flow of x.

$$\begin{aligned} L_X Y &= \text{``}\frac{\partial}{\partial t} Y(\varphi_t(p))\Big|_{t=0} \text{''}, Y(\varphi_t(p)) \in T_{\varphi_t(p)} M \\ &= \frac{\partial}{\partial t} D_{\varphi_t(p)} \varphi_{-t}(Y(\varphi_t(p)))\Big|_{t=0} \\ &= \lim_{t \to 0} \frac{D_{\varphi_t(p)} \varphi_{-t}(Y(\varphi_t(p))) - Y(p)}{t} \end{aligned}$$

Define $g(t,x) = \displaystyle\int_0^1 f'(ts,x)\,\mathrm{d}s$, where $f'(u,x) = \dfrac{\partial f}{\partial u}$ and $f(u,x) = f(\varphi_u(x))$, for any $f \in \mathcal{C}^\infty(M)$.

$$\begin{aligned} tg(t,x) &= \int_0^1 f'(ts,x) \cdot t \cdot \mathrm{d}s = \int_0^t f'(u,x)\,\mathrm{d}u, \text{where } u = ts \\ &= f(t,x) - f(0,x) = f(t,x) - f(x) \end{aligned}$$

$\Rightarrow f(t,x) = f(x) + tg(t,x)$, $f \circ \varphi_{-t} = f(-t,x)$.

Claim 8.7. $g(0,x) = (L_X f)(x)$.

Proof. $g(0,x) = \lim\limits_{t\to 0} g(t,x) = \lim\limits_{t\to 0} \dfrac{1}{t}(f(t,x) - f(x)) = (L_X f)(x)$. □

Theorem 8.8. $L_X Y = [X,Y]$, $\forall X, Y \in \mathfrak{X}(M)$.

Proof. Using the isomorphism of $\mathfrak{X}(M)$ and $\mathrm{Der}(\mathcal{C}^\infty(M))$, we need to prove

$$L_{L_X Y} f = L_{[X,Y]} f, \qquad f \in \mathcal{C}^\infty(M)$$

Let φ_t be the flow of X and $f(t,x) = f(\varphi_t(x))$ with $g(0,x) = L_X f$.
$= f(x) + tg(t,x)$

$L_Y L_X f = L_Y g(0,-) = \lim\limits_{t\to 0} L_Y g(t,-)$.

$Z_t = \dfrac{1}{t}(D_{\varphi_t(p)}\varphi_{-t}(Y(\varphi_t)(p)) - Y(p))$, so that

$$\begin{aligned} L_X Y &= \lim_{t\to 0} Z_t \\ L_{L_X Y} f &= \lim_{t\to 0} L_{Z_t f} = \lim_{t\to 0} \frac{1}{t}(L_{D_{\varphi_t}\varphi_{-t}(Y)} f - L_Y f) \\ &= \lim_{t\to 0} \frac{1}{t}(L_{Y(\varphi_{-t}(-))}(f \circ \varphi_{-t}) - L_Y f) \\ &= \lim_{t\to 0} \frac{1}{t}(L_{Y(\varphi_t(p))}(f - tg_{-t}) - L_{Y(p)} f) \\ &= \lim_{t\to 0} \frac{1}{t}(L_{Y(\varphi_t(p))}(f - tg_{-t}) - L_{Y(p)} f) \\ &= \lim_{t\to 0} \frac{1}{t}(L_{Y(\varphi_t(-))} f - L_{Y(-)} f) - \lim_{t\to 0} L_{Y(\varphi_t(-))} g_{-t} \\ &= L_X L_Y f - L_Y L_X f = L_{[X,Y]} f \end{aligned}$$

□

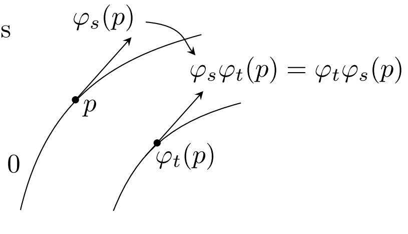
Theorem 8.9. Let $X, Y \in \mathfrak{X}(M)$, φ_t, φ_s flows for x respectively Y. Then $[X,Y] \equiv 0 \Leftrightarrow \varphi_t \circ \varphi_s = \varphi_s \circ \varphi_t$, $\forall s, t$.

Proof.

"$\Leftarrow$"

φ_t, φ_s commuting means that φ_t maps flowlines of Y to flowlines of Y.
$\Rightarrow D\varphi_t(Y) = Y$.

$[X,Y] = L_X Y = \lim\limits_{t\to 0} \dfrac{1}{t}(D\varphi_t(Y) - Y) = 0$



"⇒" For $p \in M$, consider

$$v(t) = D_{\varphi_t(p)}\varphi_{-t}Y(\varphi_t(p)) \in T_pM$$
$$\dot{v}(t) = \frac{\partial}{\partial t}v(t)\bigg|_{t=0} = (L_XY)(p) = [X,Y](p) = 0$$

Take $p = \psi_s(q)$, then $\dfrac{\partial p}{\partial s} = Y$.

$$\frac{\partial}{\partial s}\varphi_t(p) = (D\varphi_t)\left(\frac{\partial p}{\partial s}\right) = D\varphi_t(Y) = Y$$

since $v(t)$ is independent of t. So $\varphi_t(\psi_s(q))$ is a flowline of Y starting at $p = \varphi_t(q)$ at time $s = 0$.

By the uniqueness of the flowline of Y through p, we have

$$\varphi_t\psi_s(q) = \psi_s(p) = \psi_s(\varphi_t(q))$$

$\varphi_t \circ \psi_s = \psi_s \circ \varphi_t$ whenever both sides are defined.

□

Theorem 8.10. Let $X_1, \ldots, X_k \in \mathfrak{X}(M)$, s.t. $[X_i, X_j] \equiv 0$ for all i, j, and $X_1(p), \ldots, X_k(p)$ are linearly independent in T_pM for all $p \in M$. Then around every point $p \in M$, there is a chart (U, φ), such that $D_q\varphi(X_i(q)) = \dfrac{\partial}{\partial X_i}$ for all i and all $q \in U$.

Proof. The problem is local, so we may assume M is $\mathbb{R}^n$.

After a linear change of basis for $\mathbb{R}^n$, we may assume $X_i(0) = \dfrac{\partial}{\partial x_i}$ for $i \in \{1, \ldots, k\}$. So $X_1(0), \ldots, X_k(0), \dfrac{\partial}{\partial x_{k+1}}, \ldots, \dfrac{\partial}{\partial x_n}$ is a basis for $\mathbb{R}^n = T_0\mathbb{R}^n$. $\exists$ open neighborhood U of 0 in $\mathbb{R}^n$ and an $\varepsilon > 0$, s.t. the local flows φ^i of X_i are defined for all $(p,t) \in U \times (-\varepsilon, \varepsilon)$. Define $f : U \to \mathbb{R}^n$ by

$$f(x_1, \ldots, x_n) = \varphi^1_{X_1} \circ \varphi^2_{X_2} \circ \cdots \circ \varphi^k_{X_k}(0, \ldots, 0, x_{k+1}, \ldots, x_n)$$

Without loss of generality, this is defined for all $(x_1, \ldots, x_n) \in U$. By the assumption $[X_i, X_j] \equiv 0$, the φ^i and φ^j commute.

f is smooth and

$\dfrac{\partial f}{\partial x_i}(0) = X_i(0)$ for $i \in \{1, \ldots, k\}$, We also have $\dfrac{\partial f}{\partial x_i}(x) = X_i(x)$

$\dfrac{\partial f}{\partial x_i}(0) = \dfrac{\partial}{\partial x_i}$ for all i

$f(0) = 0$

$f(0) = 0$.

$D_0 f\left(\dfrac{\partial}{\partial x_i}\right) = \dfrac{\partial}{\partial x_i}$ for $i \geqslant k+1$.

For any $x \in U$, we have $D_x f\left(\dfrac{\partial}{\partial_i}\right) = X_i(f(x))$ for $i \leqslant k$.

If U is small enough, then $f : U \to f(U)$ is a diffeomorphism. Define $\varphi := f^{-1}$, $D_p\varphi(X_i) = \dfrac{\partial}{\partial x_i}$ for all $p \in f(U)$. □

Chapter 9

The Frobenius Theorem

9.1 Integral Submanifold

If M^n is decomposed into k-dimensional manifolds $L^k \subset M$ which are the image of injective immersions, the $i : L \hookrightarrow M$ and $(D_p i)(T_p L)$ is a k-dimensional subspace of $T_{i(p)}M$.

Suppose that $E \subset TM$ is a rank k subbundle.

Definition 9.1. A submanifold $S \overset{i}{\subset} M$ is called an **integral submanifold** for E if $\forall p \in S$,

$$(D_p i)(T_p S) \subset E_p$$

Definition 9.2. E is called **integrable** if through every point $p \in M$, there is a k-dimensional integral submanifold for E.

9.2 The Frobenius Theorem

Theorem 9.1 (Frobenius Theorem)**.** For a rank k subbundle $E \subset TM$, the following are equivalent:

(1) E is integrable.

(2) $\Gamma(E)$ is closed under $[\ ,\ ]$.

(3) there is an atlas (U_i, φ_i) for M, $i \in I$, such that $\forall p \in U_i$,

$$(D_p \varphi_i)(E_p) \ni \frac{\partial}{\partial x_j} \qquad \text{for } j \in \{1, \dots, k\}$$

Proof. (3)$\Rightarrow$(1): Let (U, φ) be a chart as in (3). In $\varphi(U)$, the slices given by

$$x_{k+1} = c_{k+1}, \dots, x_n = c_n$$

are k-dimensional integral submanifold of $D\varphi(E)$. Applying φ^{-1}, we obtain k-dimensional integral submanifold for $E\Big|_U$.

(1)⇒(2): Let $L \subset M$ be a k-dimensional integral submanifold for E through $p \in M$. If $X, Y \in \Gamma(E)$, then there exist unique $\tilde{X}, \tilde{Y} \in \mathfrak{X}(L)$, s.t.

$$D_i(\tilde{X}) = X\Big|_{i(L)}$$

$$D_i(\tilde{Y}) = Y\Big|_{i(L)}$$

$$[X, Y](p) = [D_i(\tilde{X}), D_i(\tilde{Y})](p) = (D_i[\tilde{X}, \tilde{Y}])(p) \in E_p$$

The second equlity is by the following claim:

Claim 9.2. $f : M \to N$ is a smooth map, $X, Y \in \mathfrak{X}(M)$.

$$D_p f([X, Y](p)) = [Df(X), Df(Y)](f(p))$$

Proof. Let $h \in \mathcal{C}^\infty(N)$.

$$\begin{aligned}(L_{Df(X)}h)(q) &= D_q h(D_p f(X(p))) \\ &= D_p(h \circ f)(X(p)) \\ &= (L_X(h \circ f))(p)\end{aligned}$$

Note that $q = f(p)$. So

$$(L_{Df(X)}h) \circ f = L_X(h \circ f)$$

Then

$$\begin{aligned}L_{[Df(X),Df(Y)]}h &= L_{Df(X)}L_{Df(Y)}h - L_{Df(Y)}L_{Df(X)}h \\ &= L_X((L_{Df(Y)}h) \circ f) - L_Y((L_{Df(X)}h) \circ f) \\ &= L_X L_Y(h \circ f) - L_Y L_X(h \circ f) \\ &= L_{[X,Y]}(h \circ f) \\ &= L_{Df[X,Y]}h\end{aligned}$$

Thus,

$$[Df(X), Df(Y)] = Df[X, Y]$$

□

(2)⇒(3): Proving (3) is a local problem, so we may work on an open neighborhood U of 0 in $\mathbb{R}^n$.

Step 1: Consider the projection
$$\begin{aligned}\pi : U &\to \mathbb{R}^k \\ (x_1, \dots, x_n) &\mapsto (x_1, \dots, x_k)\end{aligned}$$

Suppose that $D_0\pi\Big|_{E_0}$ is an isomorphism. Then we may assume $D_p\pi\Big|_{E_p}$ is an isomorphism for all $p \in U$.

Step 2: After a linear change of coordinates on $\mathbb{R}^n$, we may assume that $D_0\pi\Big|_{E_0}$ is an isomorphism. By Step 1, the same is then true for all $p \in U$.

Step 3: Let U and π be as above. Fix $z_i \in \Gamma(E\big|_U)$, so that

$$D\pi(z_i) = \frac{\partial}{\partial x_i} \qquad \text{for } i \in \{1, \dots, k\}$$

Then $z_1(p), \dots, z_k(p)$ are a basis of E_p for every $p \in U$. By (2), we have $[z_i, z_j] \in \Gamma(E)$. Then

$$D\pi[z_i, z_j] = [D\pi(z_i), D\pi(z_j)] = \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$$

By injectivity of $D\pi\Big|_E$, we conclude $[z_i, z_j] = 0$.

Step 4: Since z_i pairwise commute, there are local coordinates, s.t. $z_i = \dfrac{\partial}{\partial x_i}$. ☐

9.3 Foliation

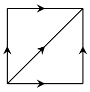
Definition 9.3. Let M be a smooth n-dimensional manifold, $0 \leqslant k \leqslant n$.
A k-dimensional **foliation** $\mathcal{F}$ of M is a decomposition of M into k-dimensional injectively immersed manifolds which is locally trivial in the following sense: $\forall p \in M$, $\exists$ open neighborhood U and a diffeomorphism $\varphi : U \to \mathbb{R}^n$, s.t. the intersections of injective immersed manifolds making up $\mathcal{F}$ with U are mapped by φ to the slices

$$x_{k+1} = c_{k+1}, \dots, x_n = c_n$$

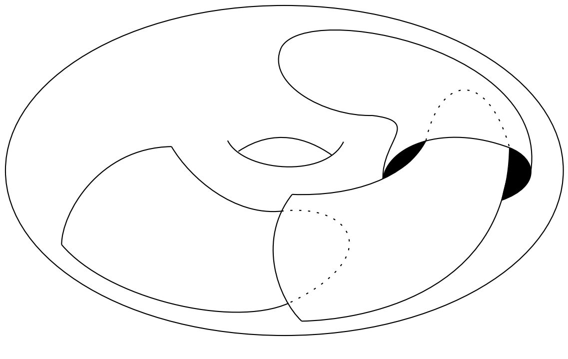
A subbundle $E \subset TM$ is integrable if and only if E consists of vectors tangent to the leaves of a foliation, this is true if and only if $\Gamma(E)$ is closed under [,].

Example 9.1. Every rank 1 subbundle $E \subset TM$ is integrable to a 1-dimensional foliation.

Example 9.2. $k = 2$, locally $E = \operatorname{span}\{x, y\}$. Then
Integrate x to get 1-dimensional integral submanifold for E.
Integrate y to get 1-dimensional integral submanifold for E.

Example 9.3. $M = T^2 = S^1 \times S^1 =$  $= ([0,1] \times [0,1])/\sim$

$E \subset TT^2$ spanned by $a\dfrac{\partial}{\partial x} + b\dfrac{\partial}{\partial y} = X$
If $a/b \in \mathbb{Q}$, then all flowlines of X are periodic, so $= S^1$.
If $a/b \notin \mathbb{Q}$, then all flowlines of X are $\cong \mathbb{R}$, and are dense in T^2.
Let $T_i = S^1 \times D^2$, then $S^3 = \mathbb{R}^2 \cup \{\infty\} = T_1 \cup T_2$.



Reeb Foliation of S^3

Chapter 10

Differential Forms and Multilinear Algebra

10.1 Differential Forms

M is a smooth manifold, $\dim M = n$.

Definition 10.1. A differential form of degree k, or a k-form, is a $\mathcal{C}^\infty(M)$-multilinear map

$$\begin{aligned}\omega : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) &\to \mathcal{C}^\infty(M)\\ (X_1, \ldots, X_k) &\mapsto \omega(X_1, \ldots, X_k)\end{aligned}$$

with the property

$$\omega(X_{\sigma(1)}, \ldots, X_{\sigma(k)}) = \operatorname{sign}(\sigma) \cdot \omega(X_1, \ldots, X_k)$$

$\operatorname{sign}(\sigma) = \pm 1$ according to whether the number of transpositions in σ is even or odd.

Lemma 10.1. $\omega(X_1, \ldots, X_k)(p)$ depends on X_i only through $X_i(p) \Rightarrow \omega(p) : T_pM \times T_pM \to \mathbb{R}$ is k-multilinear.

$$\omega(p)(X_{\sigma(1)}(p), \ldots, X_{\sigma(k)}(p)) = \operatorname{sign}(\sigma)\omega(p)(X_1(p), \ldots, X_k(p)), \qquad \omega \in \Gamma(\Lambda^k)$$

Proof. We only have to prove the Lemma for $i = 1$.

Step 1: Suppose there is an open set $U \subset M$, s.t. $X_1\Big|_U \equiv 0$. Let $\rho : M \to \mathbb{R}$ be a smooth bump function with $\rho(p) = 1$ for a fixed $p \in U$ and $\operatorname{supp}(\rho) \subset U$. Then $\rho \cdot X_1 \equiv 0$.

$$0 = \omega(\rho X_1, X_2, \ldots, X_k) = \rho \cdot \omega(X_1, \ldots, X_k) \ \Rightarrow\ \omega(X_1, \ldots, X_k)(p) = 0$$

Step 2: Suppose $p \in M$ is such that $X_1(p) = 0$. Using a chart (U, φ) around p, we can write

$$X_1\Big|_U = \sum_{j=1}^n f_j \frac{\partial}{\partial x_j}, \quad f_j \in \mathcal{C}^\infty(U)$$

Let $\rho : U \to \mathbb{R}$ be a $\mathcal{C}^\infty$ bump function with $\rho\Big|_V \equiv 1$ for $V \subset U$ a smalleer neighborhood of p and $\mathrm{supp}(\rho) \subset U$. Then $\rho \cdot f_j \in \mathcal{C}^\infty(M)$ and $\rho \cdot f_j\Big|_V \equiv f_j$. Similarly, $Y_i = \rho \cdot \frac{\partial}{\partial x_j} \in \mathfrak{X}(M)$ and $Y_j\Big|_V \equiv \frac{\partial}{\partial x_j}$. Then $Y := \sum_{j=1}^n (\rho \cdot f_j) \cdot Y_j \in \mathfrak{X}(M)$ has the property that $Y\Big|_V \equiv X_1\Big|_V \Rightarrow (X_1 - Y)\Big|_V \equiv 0$, so by Step 1:

$$\omega(X_1 - Y, X_2, \dots, X_k)(p) = 0$$

$$0 = \omega(X_1 - Y, X_2, \dots, X_k)(p) = \omega(X_1, \dots, X_k)(p) - \omega(Y, X_2, \dots, X_k)(p)$$

$$\omega(Y, X_2, \dots, X_k)(p) = \sum_{j=1}^n \underbrace{(\rho \cdot f_j)(p)}_{=0,\text{ because } f_j(p)=0 \text{ since } X_1(p)=0} \omega(Y_j, X_2, \dots, X_k)(p) = 0$$

$$\Rightarrow \qquad \omega(X_1, \dots, X_k)(p) = 0 \text{ wherever } X_1(p) = 0$$

Step 3: Suppose $X_1, X_1' \in \mathfrak{X}(M)$ with $X_1(p) = X_1'(p)$. Then applying Step 2 to $X_1 - X_1'$, we see

$$\omega(X_1, \dots, X_k)(p) = \omega(X_1', X_2, \dots, X_k)(p)$$

□

10.2 Excursion into Multilinear Algebra

Let V, W be (finite-dimensional) $\mathbb{R}$-vector spaces.

Definition 10.2. A **tensor product** for V and W is a bilinear map

$$\begin{tikzcd}
\varphi : V \times W \arrow[r] \arrow[d, "f"'] & T \arrow[dl, "\exists! \overline{f} \text{ linear}"] \\
Z &
\end{tikzcd}$$

(Universal property of tensor product), where T is a $\mathbb{R}$-vector space, such that every bilinear map $f : V \times W \to Z$ factorizes uniquely through φ.

Theorem 10.2. A tensor product exists, and is unique up to unique isomorphism.

Proof. Uniqueness: Suppose $\varphi_i : V \times W \to T_i$, $i = 1, 2$ are two tensor products satisfying the universal property.

$$\begin{tikzcd}
V \times W \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & T_1 \arrow[dl, "\overline{\varphi}_2 \text{ linear}"] \\
T_2 &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
V \times W \arrow[r, "\varphi_2"] \arrow[d, "\varphi_1"'] & T_1 \arrow[dl, "\overline{\varphi}_1 \text{ linear}"] \\
T_2 &
\end{tikzcd}$$

$$\begin{aligned} \varphi_2 &= \overline{\varphi}_2 \circ \varphi_1 \\ &= \overline{\varphi}_2 \circ \overline{\varphi}_1 \circ \varphi_2 \end{aligned} \qquad \text{and similarly} \qquad \begin{aligned} \varphi_1 &= \overline{\varphi}_1 \circ \varphi_2 \\ &= \overline{\varphi}_1 \circ \overline{\varphi}_2 \circ \varphi_1 \end{aligned}$$

$$\begin{tikzcd} V \times W \arrow[r, "\varphi_2"] \arrow[d, "\varphi_2"'] & T_2 \arrow[dl, "\overline{\varphi}_2 \circ \overline{\varphi}_1 = \mathrm{Id}_{T_2}"] \\ T_2 & \end{tikzcd}$$

Similarly $\overline{\varphi}_1 \circ \overline{\varphi}_2 = \mathrm{Id}_{T_1}$

$\Rightarrow$ the $\overline{\varphi}_i$ are isomorphisms inverse to each others.
These are the only choices of isomorphisms between the T_i, which make the triangle commute.

Existence: Let X be the $\mathbb{R}$-vector space with basis $V \times W$. Let $Y \subset X$ be the subspace generated by elements in X of the form:

$$(av_1 + bv_2, w) - a(v_1, w) - b(v_2, w) \text{ and } (v, aw_1 + bw_2) - a(v, w_1) - b(v, w_2)$$

where $T := X/Y$ is the quotient vector space. The coset of (v, w) will be denoted $v \otimes w$. Define $\varphi : V \times W \to T$ by

$$(v, w) \mapsto v \otimes w$$

Claim 10.3. (T, φ) is a tensor porduct of V and W.

Proof. 1. φ is bilinear

$$\begin{aligned} \varphi(av_1 + bv_2, w) &= (av_1 + bv_2) \otimes w \\ &= av_1 \otimes w + av_2 \otimes w \end{aligned}$$

So φ is linear in the first argument. Similar argument for the second argument.

2. Given a bilinear $f : V \times W \to Z$, define $\overline{f}(v \otimes w) := f(v, w)$, and extended linearly to T. Then $\overline{f} : T \to Z$ is a well-defined linear map. Moreover, $\overline{f} \circ \varphi(v, w) = \overline{f}(v \otimes w) = f(v, w)$, so $f = \overline{f} \circ \varphi$.

3. Given f, the $\overline{f}$ in 2 is unique. Suppose $g : T \to Z$ is any linear map with $f = g \circ \varphi$. Then

$$\overline{f}(v \otimes w) = f(v, w) = g(v \otimes w)$$

Since the $v \otimes w$ span T, we conclude $\overline{f} \equiv g$. □

□

From now on, we write $T = V \otimes W$ and $\varphi(v, w) = v \otimes w$ for the unique tensor product of V and W.

Suppose $v_1, \dots, v_n \in V$ and $w_1, \dots, w_m \in W$ are basis of V respectively W. Then $v_i \otimes w_j$, $i \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$ is a basis of $V \otimes W$.

$$\dim(V \otimes W) = \dim V \cdot \dim W$$

$$\begin{tikzcd} V \times W \arrow[r, "\varphi"] \arrow[d, "f"'] & V \otimes W \arrow[dl, "\overline{f}"] \\ \mathbb{R} & \end{tikzcd}$$

The space of bilinear maps from $V \times W$ to $\mathbb{R}$ is $(V \otimes W)^*$.

If $V_1, \dots, V_k$ are finite-dimensional $\mathbb{R}$-vector spaces, there is a unique tensor product $V_1 \otimes \cdots \otimes V_k$ which has the universal property for k-linear maps:

$$\begin{tikzcd}
V_1 \times \cdots \times V_k \arrow[r, "\varphi"] \arrow[d, "f"'] & V_1 \otimes \cdots \otimes V_k \arrow[dl, "\exists! \overline{f} \text{ linear}"] \\
Z &
\end{tikzcd}$$

For a single $\mathbb{R}$-vector space V denoted

$$T^k(V) = \underbrace{V \otimes \cdots \otimes V}_{k \text{ factors}}$$

Let $T^0(V) = \mathbb{R}$ and $T^1(V) = V$, then the tensor algebra of V is

$$T(V) = T^*(V) = \bigoplus_{k=0}^{\infty} T^k(V)$$

The multiplication in this algebra is induced by

$$v_1 \otimes v_2 \otimes \cdots \otimes v_k \cdot w_1 \otimes w_2 \otimes \cdots \otimes w_l = v_1 \otimes \cdots \otimes v_k \otimes w_1 \otimes \cdots \otimes w_l, \qquad v_i, w_j \in V$$

Then $\cdot$ is written $\otimes$ and $T^k(V) \times T^l(V) \xrightarrow{\otimes} T^{k+1}(V)$.

10.3 Exterior Algebra

$$\begin{tikzcd}
V \times \cdots \times V \arrow[r] \arrow[d, "f"'] & T^k(V) \arrow[dl, "\exists! \overline{f}"] \\
Z &
\end{tikzcd}$$

, where k is multilinear.

Consider only skew-symmetric f, so that

$$f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \operatorname{sign}(\sigma) \cdot f(v_1, \dots, v_k)$$

$$\begin{tikzcd}
V \times \cdots \times V \arrow[r, "\varphi"] \arrow[d, "f"'] & \Lambda^k(V) \arrow[dl, "\exists! \overline{f}"] \\
Z &
\end{tikzcd}$$

$$T^*(V) = \bigoplus_{k \geqslant 0} T^k(V)$$

$\cup$

A the ideal generated by $v_1 \otimes v_2 + v_2 \otimes v_1, v_i \in V$ the “alternating ideal”

$\parallel$

$\bigoplus_{k \geqslant 0} A^k$ where $A^k = A \cap T^k(V)$

$$\begin{aligned}
A^0 &= 0 \\
A^1 &= 0 \\
A^2 &= \operatorname{span}\{v_1 \otimes v_2 + v_2 \otimes v_1 \mid v_i \in V\} \\
&\cap \\
T^2(V) &= \operatorname{span}\{v_1 \otimes v_2 \mid v_i \in V\} \\
A^k &= \operatorname{span} \bigcup_{p+q+2=k} T^q(V) \otimes A^2 \otimes T^p(V)
\end{aligned}$$

Definition 10.3. $T^*(V)/A := \Lambda^*(V)$ is the **exterior algebra** of V.

Suppose f is skew-symmetric. Then

$$f(v_1, v_2) = -f(v_2, v_1)$$

$$\begin{aligned} \Rightarrow \qquad & \overline{f}(v_1 \otimes v_2) = -\overline{f}(v_2 \otimes v_1) \\ \Leftrightarrow \qquad & \overline{f}(v_1 \otimes v_2 + v_2 \otimes v_1) = 0 \end{aligned}$$

Lemma 10.4. A k-multilinear map $f : V \times \cdots \times V \to Z$ is skew-symmetric if and only if $\overline{f}\Big|_{A^k} \equiv 0$.

f is k-multilinear and skew-symmetric

$$\begin{tikzcd} V \times \cdots \times V \arrow[r, "\varphi"] \arrow[d, "f"'] & T^k(V) \arrow[r, "\pi"] \arrow[dl, "\overline{f}"'] & T^k(V)/A^k = \Lambda^k(V) \arrow[dll, "\overline{\overline{f}}"] \\ Z & & \end{tikzcd}$$

$$\Lambda(V) = \bigoplus_{k \geqslant 0} \Lambda^k(V)$$

Let $\dim V = n$, and $v_1, \ldots, v_n$ is a basis of V. Then $v_{i_1} \otimes \cdots \otimes v_{i_k}$, $i_j \in \{1, \ldots, n\}$ form a basis for $T^k(V)$. And $\dim T^k(V) = n^k$.

$$[v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}] = \text{sign}(\sigma)[v_1 \otimes \cdots \otimes v_k] \text{ in } \Lambda^k(V)$$

If we have two repeating indices, we are going to have zero, because $v_1 \otimes v_1 +$ ~~$v_1 \otimes v_1$~~ $\in A^2$. So if you have two indices which are the same, then the corresponding elements in the exterior algebra is zero. For those the indices are different, then you can use this equation to just put them in a sending order, whatever their order have here, up to sign, it is just this. Then we are done.

$$\begin{array}{ll} [v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}] \ , & 1 \leqslant i_1 < i_2 < \cdots \leqslant n \text{ form a basis for } \Lambda^k(V) \\ \qquad \qquad \| & \\ v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k} & \end{array}$$

So we think of the exterior algebra as the quotient of the tensor algebra, we don't usually write elements in this quotient as cosets this bracket, we just write like this. It says

$$\dim \Lambda^k(C) = \binom{n}{k}$$

That specify all the spaces. So in particular,

$$\Lambda^k(V) = 0 \text{ if } k > n$$

So this graded algebra actually stops after the degree n. That was not the case for the tensor. The tensor algebra has arbitrary many elements and tesor algebra has a vector space over $\mathbb{R}$ is infinite dimension. But since the spaces vanish in the degree larger than n for the exterior algebra and these space are finite dimensional, the whole exterior algebra is finite dimension. So

$$\dim \Lambda^*(V) = \sum_{k=0} \binom{n}{k} = (1+1)^n = 2^n$$

Let us consider something about the induced map of tensor products or exterior products. This is kind of functoriality properties of this instructions. First of all, suppose $f_i : V_i \to W_i$ are linear maps.

$$\begin{aligned} &V_1 \otimes V_2 \xrightarrow{f} W_1 \otimes W_2 \text{ is a linear map} \\ &v_1 \otimes v_2 \mapsto f_1(w_1) \otimes f_2(v_2) \end{aligned}$$

where $v_1 \otimes v_2$ are called decomposable elements of $V_1 \otimes V_2$. What we do is we've constructed the tensor product. It is obviously spaned by these decomposable elements. Then the general element is not decomposable, but it is a linear combination of decomposable elements. Because the decomposable once are spanning set, you can make of this definition. Same thing works for the exterior algebra, if V_1 is the same as V_2. Using these constructions, every linear map $f : V \to W$ induces an algebra homomorphism

$$\begin{aligned} T(f) &: T^*(V) \to T^*(W) \\ v_1 &\otimes \cdots \otimes v_k \mapsto f(v_1) \otimes \cdots f(v_k) \end{aligned}$$

Similarly,

$$\begin{aligned} &\Lambda(f) : \Lambda^*(V) \to \Lambda^*(W) \text{ is an algebra homomorphism} \\ &v_1 \wedge \cdots \wedge v_k \mapsto f(v_1) \wedge \cdots \wedge f(v_k) \end{aligned}$$

What has this happened to do with determinant? Let $\dim V = n$ and $f : V \to V$ is linear, then

$$\Lambda^n(f) : \Lambda^n(V) \to \Lambda^n(V)$$

where $\Lambda^n(V)$ is 1-dimensional.

Claim 10.5. $\Lambda^n(f)$ is multiplication by $\det(f)$.

$$\begin{tikzcd} V \times \cdots \times V \arrow[r] \arrow[d, "f"'] & \Lambda^k(V) \arrow[dl, "\overline{f}"] \\ \mathbb{R} & \end{tikzcd}$$

where f is k-linear nad skew-symmetric. The space of k-linear skew-symmetric maps $f : V \times \cdots V \to \mathbb{R}$ is naturally $(\Lambda^k(V))^* = \Lambda^k(V^*)$. Then $\lambda_1 \wedge \cdots \wedge \lambda_k \in \Lambda^k(V^*)$ acts as a linear map $\Lambda^k(V) \to \mathbb{R}$ by $(\lambda_1 \wedge \cdots \wedge \lambda_k)(v_1 \wedge \cdots \wedge v_k) = \sum_\sigma \operatorname{sign}(\sigma)\lambda_1(v_{\sigma(1)}) \cdots \lambda_k(v_\sigma(k)) \in \mathbb{R}$. For instance,

$k = 2$ $\qquad (\lambda_1 \wedge \lambda_2)(v_1 \wedge v_2) = \lambda_1(v_1)\lambda_2(v_2) - \lambda_1(v_2)\lambda_2(v_1)$

10.4 Multilinear Vector Bundle Theory

If $\omega : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to \mathcal{C}^\infty(M)$ is a differential form of degree k on M, then

$$\omega(p) : T_pM \times \cdots \times T_pM \to \mathbb{R}$$

is well-defined, k-linear and skew-symmetric.

$$\Rightarrow \qquad \omega(p) \in \Lambda^k T_p^* M$$

Instead of saying for $\Lambda^k T^* M$, more generally, that in fact the multilinear construction we have done are extended from vector spaces to vector spaces. Vector space is a vector bundle over the points. To replace the points by arbitrary manifold, essentially, everything was the same. As an example, we will define tensor for vector bundles. Let $E \xrightarrow{\pi_E} M$, $F \xrightarrow{\pi_F} M$ be two smooth vector bundles of rank k and l, respectively. We can find an open covering $\{U_i \mid i \in I\}$ of M, so that on each U_i, both E and F are trivial.

$$\varphi_i : \pi_E^{-1}(U_i) \to U_i \times \mathbb{R}^k$$
$$\psi_i : \pi_F^{-1}(U_i) \to U_i \times \mathbb{R}^l$$

where these are local trivialization. Now the question is do this local definitions fit together properly, you have something is well-defined independently to your local trivialization? $U_i \times (\mathbb{R}^k \otimes \mathbb{R}^l)$ represents $E \otimes F$ over U_i. If $U_i \cap U_j \neq \varnothing$, then

$$\begin{aligned} \varphi_j \circ \varphi_i^{-1} : (U_i \cap U_j) \times \mathbb{R}^k &\to (U_i \cap U_j) \times \mathbb{R}^k \\ (p, v) &\mapsto (p, g_{ij}(p) \cdot v) \end{aligned}$$

where $g_{ij} : U_i \cap U_j \to GL_k(\mathbb{R})$. Similarly,

$$\psi_j \circ \psi_i^{-1}(p, v) = (p, f_{ji}(p) \cdot v)$$

for smooth $f_{ij} : U_i \cap U_j \to GL_l(\mathbb{R})$. Consider $g_{ji} \otimes f_{ji} : U_i \cap U_j \to GL_{k \cdot l}(\mathbb{R})$

$$p \mapsto g_{ji}(p) \otimes f_{ji}(p)$$

by

$$(g_{ji}(p) \otimes f_{ji}(p))(v \otimes w) = g_{ji}(p)(v) \otimes f_{ji}(p)(w)$$

where $g_{**} \otimes f_{**}$ is a cocycle, and $E \otimes F$ is the corresponding vector bundle of rank $k \cdot l$, trivial over each U_i. This is how using cocycles to define make precise that the vector bundle $E \otimes F$ is the fibrewise tensor product of the fibres E and F. Fibres of E and F form the vector spaces and over every point is just take the tensor product of the fibre.

Now we want to extend this and we are not doing with for the tensor algebra, because tensor algebra is infinite dimension and we don't want to speak of infinite rank vector bundles.

Given a single vector bundle $E \to M$, we can use this construction to define $T^m(E) \to M$ for every $m \geqslant 0$. This descends to a definition of $\Lambda^m(E) \to M$ by taking the quotient bundle $T^m(E)/A^m$.

For example, let E, F be vector bundles over M, and $f : E \to F$ a homomorphism of vector bundles. Then

$$T^m(f) : T^m(E) \to T^m(F)$$
$$\Lambda^m(f) : \Lambda^m(E) \to \Lambda^m(F)$$

are also homomorphism of vector bundles. I said the differential forms has a value a the point which is an element of $\Lambda^k T^* M$. Now we have constructed this vector bundle and apply this to the cotangent bundle.

$$\begin{tikzcd}
f^*E \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\
M \arrow[r, "f"'] & N
\end{tikzcd}$$

$$f^*\Lambda^m(E) = \Lambda^m(f^*E)$$

We have now defined $\Lambda^k T^* M$, and $\Gamma(\Lambda^k T^* M)$ are differential forms of degree k on M.

Now we want to apply the above discussion to differential forms. Suppose $f : M \to N$ is a smooth map. Then $f^*(\Lambda^k T^* N)$ is a vector bundle over M. If $\omega \in \Gamma(\Lambda^k T^* N) = \Omega^k(N)$ is a k-form on N, then we define $f^*\omega$ as follows

$$(f^*\omega)(X_1, \ldots, X_k)(p) = \omega(f(p))(D_p f(X_1), \ldots, D_p f(X_k))$$

This is a k-form on M.

$$\begin{tikzcd}
TM \arrow[r, "Df"] \arrow[dr] & f^*TM \arrow[r] \arrow[d] & TN \arrow[d] \\
& M \arrow[r, "f"'] & N
\end{tikzcd}$$

$$(Df)^* : \underbrace{(f^*TN)^*}_{=f^*T^*N} \to T^*N$$

$$\Lambda^k((Df)^*) : \underbrace{\Lambda^k f^* T^* N}_{=f^*\Lambda^k T^* N} \to \Lambda^k T^* M$$

Now we can say the following:

$$(f^*(\omega))(p) = \Lambda^k (D_p f)^* \omega(f(p))$$

where the derivative of p at f is a linear map

$$D_p f : T_p M \to T_{f(p)} N$$
$$(D_p f)^* : T^*_{f(p)} N \to T^*_p M$$

Here we doing this not from the cotangent space, but on the exterior products.

Chapter 11

Integration of Forms

11.1 Orientation

To discuss the application of the smooth linear algebra of vector bundles, we have the following proposition.

Proposition 11.1. Suppose $E \xrightarrow{\pi} M$ is a smooth vector bundle of rank k. Then the following are equivalent:

(1) E is orientable.

(2) $\Lambda^k E$ is orientable.

(3) $\Lambda^k E$ is trivial.

Proof. $\Lambda^k E$ has rank $\begin{pmatrix} k \\ k \end{pmatrix} = 1$.

(2)⇔(3): we proved before for arbitrary rank 1 bundle.

(1)⇔(2): By definition, E is orientable if and only if $\exists$ system of local trivializations (U_i, φ_i),

$$\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^k$$

for which all $\varphi_j \circ \varphi_i^{-1}$ are orientation-preserving on $\{p\} \times \mathbb{R}^k$ for all $p \in U_i \cap U_j$.

$$\begin{aligned} \varphi_j \circ \varphi_i^{-1} : (U_i \cap U_j) \times \mathbb{R}^k &\to (U_i \cap U_j) \times \mathbb{R}^k \\ (p, v) &\mapsto (p, g_{ji}(p) \cdot v) \end{aligned}$$

where $g_{ji} : U_i \cap U_j \to GL_k(\mathbb{R})$. So all g_{ji} takes value in $GL_k^+(\mathbb{R}) \subset GL_k(\mathbb{R}) \Leftrightarrow \det g_{ji}(p) > 0$ for all $p \in U_i \cap U_j$. $(U_i, \Lambda^k \varphi_i)$ form a system of local trivializations for $\Lambda^k E$, whose transition maps are $\det g_{ji}(p)$. So if (1) holds, then (2) follows.

For the converse, choose an open covering of M by U_i such that both E and $\Lambda^k E$ are trivial over all U_i.

Proof. Over each U_i, we have trivializations

$$\begin{aligned} \varphi_i : \pi^{-1}(U_i) &\to U_i \times \mathbb{R}^k \qquad & \pi : E \to M \\ \psi_i : (\pi')^{-1}(U_i) &\to U_i \times \mathbb{R}^k \qquad & \pi' : \Lambda^k E \to M \end{aligned}$$

If (2) holds, we may choose the ψ_i, so that all $\psi_j \circ \psi_i^{-1}$ are orientable-preserving on $\mathbb{R}$.

$$E_p \xrightarrow{\varphi_i} \{p\} \times \mathbb{R}^k$$

$$\Lambda^k E_p \xrightarrow{\Lambda^k \varphi_i} \{p\} \times \mathbb{R}$$

$$\Lambda^k E_p \xrightarrow{\psi_i} \{p\} \times \mathbb{R}$$

By composing φ_i with a reflection in a hyperplane in $\mathbb{R}^k$, we may assume that $\Lambda^k \varphi_i$ and ψ_i define the same orientation on $\Lambda^k E_p$.

Since the ψ_i have orientation-preserving transition map by assumption, the same is now true for the φ_i, so (2)⇒(1). □

□

Remark. E^* is (non-canonically) isomorphic to E.

If $\langle \, , \, \rangle$ is a metric on E, then

$$\begin{aligned} f : E &\to E^* \\ v &\mapsto \langle v, - \rangle \end{aligned}$$

is a bundle homomorphism which is an isomorphism.

Definition 11.1. A smooth manifold M is **orientable** if $TM \to M$ is an orientable vector bundle.

Definition 11.2. A **volume form** on M is a differential form $\omega \in \Gamma(\Lambda^n T^* M)$ where $n = \dim M$, s.t. $\omega(p) \neq 0$, $\forall p \in M$.

Corollary 11.2. For a smooth n-dimensional manifold, the following are equivalent:

(1) M is orientable.

(2) $\Lambda^n TM$ is orientable.

(3) $\Lambda^n TM$ is trivial.

(4) M admits a volume form.

Let $\Omega^k(M) = \Gamma(\Lambda^k T^* M)$. When $k = 0$, $\Omega^0(M) = \Gamma(M \times \mathbb{R}) = \mathcal{C}^\infty(M)$. We define

$$\begin{aligned} d : \mathcal{C}^\infty(M) &\to \Omega^k(M) \\ f &\mapsto df \end{aligned}$$

where

$$\begin{aligned} (df)(p) : T_p M &\to \mathbb{R} \\ x &\mapsto D_p f(x) \end{aligned}$$

Lemma 11.3. Let $\varphi : M \to N$ be a differetiable map, $f \in \mathcal{C}^\infty(N)$. Then $\varphi^*(f) = f \circ \varphi$ and $\varphi^*(df) = d(\varphi^*(f))$. So $\varphi^* \circ d = d \circ \varphi^*$.

Proof. Let $X \in T_pM$.

$$\begin{aligned}(\varphi^* df)(X) &= df(D_p\varphi(X)) \\ &= (D_{\varphi(p)}f)(D_p\varphi(X)) \\ &= D_p(f \circ \varphi)(X) \\ &= D_p(\varphi^*(f))(X) \\ &= d(\varphi^*(f))(X)\end{aligned}$$

□

Let $U \subset \mathbb{R}^n$ be open, $f \in \mathcal{C}^\infty$.

$$df(X) = Df(X)$$

At every point $p \in U$, $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ form a basis of T_pU

$$X = \sum_{i=1}^n \lambda_i \frac{\partial}{\partial x_i}$$

Let $dx_1, \dots, dx_n$ be the dual basis of T_p^*M.

Claim 11.4. $df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$.

Proof. For all X, we must have $df(X) = Df(X)$. Take $X = \frac{\partial}{\partial x_i}$. Then $df\left(\frac{\partial}{\partial x_i}\right) = Df\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial f}{\partial x_i}$.

$$\left(\sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j\right)\left(\frac{\partial}{\partial x_i}\right) = \sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j \left(\frac{\partial}{\partial x_i}\right) = \frac{\partial f}{\partial x_i}$$

□

$$\omega = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} f_{i_1,\dots,i_k}\ dx_{i_1} \wedge \cdots \wedge dx_{i_k} \in \Omega^k(U)$$

Define

$$d\omega = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} (df_{i_1,\dots,i_k}) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \in \Omega^{k+1}(U)$$

Claim 11.5. $d^2 \equiv 0$, so $d(d\omega) = 0$.

Proof.

$$\begin{aligned}d(d\omega) &= d\sum_{i_j}\left(\sum_{\alpha=1}^n \frac{\partial f_{i_j}}{\partial x_\alpha} dx_\alpha\right) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &= \sum_{i_j}\sum_{\alpha,\beta=1}^n \frac{\partial}{\partial x_\beta}\left(\frac{\partial f_{i_j}}{\partial x_\alpha}\right) dx_\beta \wedge dx_\alpha \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &= \sum_{i_j}\sum_{\alpha<\beta} \cancel{\left(\frac{\partial f_{i_j}}{\partial x_\beta \partial x_\alpha} - \frac{\partial f_{i_j}}{\partial x_\alpha \partial x_\beta}\right)} dx_\beta \wedge dx_\alpha \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} = 0\end{aligned}$$

□

Lemma 11.6. Let $\varphi : M \to N$ be a differentiable map, $\omega \in \Omega^k(N)$. Let $M, N \subset \mathbb{R}^n$ be open subsets. Then $d\varphi^*\omega = \varphi^* d\omega$.

Proof. $\omega = \sum\limits_{i_j} f_{i_1,\dots,i_k} dy_{i_1} \wedge \dots \wedge dy_{i_k}$.

$$\begin{aligned} d\varphi^*\omega &= d\sum_{i_j} \varphi^*(f_{i_1,\dots,i_k})\varphi^*(dy_{i_1}) \wedge \dots \wedge \varphi^*(dy_{i_k}) \\ &= d\sum_{i_j} \varphi^*(f_{i_1,\dots,i_k})d(\varphi^* y_{i_1}) \wedge \dots \wedge d(\varphi^* y_{i_k}) \\ &= \sum_{i_j} d\varphi^*(f_{i_1,\dots,i_k}) \wedge d(\varphi^* y_{i_1}) \wedge \dots \wedge d(\varphi^* y_{i_k}) \\ &= \varphi^* \left(\sum_{i_j} df_{i_1,\dots,i_k} \wedge dy_{i_1} \wedge \dots \wedge dy_{i_k} \right) = \varphi^* d\omega \end{aligned}$$

□

Claim 11.7. If $\omega \in \Omega^k(U)$ and $\eta \in \Omega^l(U)$, then $d(\omega\wedge\eta) = d\omega\wedge\eta + (-1)^k\omega\wedge(d\eta)$.

If $k = 0$, then $\omega = f \in \mathcal{C}^\infty(U)$. This formula becomes

$$d(f\eta) = df \wedge \eta + f d\eta$$

11.2 Exterior Derivative

Definition 11.3. An **exterior derivative** on a smooth manifold M is a $\mathbb{R}$-linear map $d : \Omega^k(M) \to \Omega^{k+1}(M)$ for all k with the following properties:

(1) If $k = 0$, then $df(X) = Df(X)$.

(2) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{\deg \omega}\omega \wedge d\eta$.

(3) $d^2 = 0$.

(4) d commutes with pullback by differentiable maps.

(5) If $U \subset M$ open, then $(d\omega)\Big|_U$ depends only on $\omega\Big|_U$.

Theorem 11.8. There exists a unique exterior derivative d on smooth manifolds satisfying (1)-(5).

Proof. First uniqueness, then existence. ("0-form wedge a k-form is just 0-form (=functions) times that k-form.")

Uniqueness: On 0-forms (=functions), d is determined by (1). Let $\omega \in \Omega^k$, $k > 0$. Then by (5), we need only consider $\omega\Big|_U$ for charts (U, φ). Then

$$\omega\Big|_U = \varphi^* \sum_{i_1<\dots<i_k} f_{i_1,\dots,i_k}\, dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

$$\begin{aligned}(d\omega)\Big|_U &\overset{(5)}{=\!=} d\left(\omega\Big|_U\right) = d\left(\varphi^* \sum_{i_1<\cdots<i_k} f_{i_1,\dots,i_k}\ dx_{i_1} \wedge \cdots \wedge dx_{i_k}\right)\\ &\overset{(4)}{=\!=} \varphi^* d\left(\sum_{i_1<\cdots<i_k} f_{i_1,\dots,i_k}\ dx_{i_1} \wedge \cdots \wedge dx_{i_k}\right)\\ &= \sum_{i_1<\cdots<i_k} \varphi^*(d(f_{i_1,\dots,i_k}\ dx_{i_1} \wedge \cdots \wedge dx_{i_k}))\\ &\overset{(2)}{=\!=} \sum_{i_1<\cdots<i_k} \varphi^*((df_{i_1,\dots,i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}) + f_{i_1,\dots,i_k} d(dx_{i_1} \wedge \cdots \wedge dx_{i_k}))\\ &\overset{(2)+(3)}{=\!=\!=} \sum_{i_1<\cdots<i_k} \varphi^* \underbrace{(df_{i_1,\dots,i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k})}_{\text{uniquely determined by (1)}}\end{aligned}$$

Existence: Let $\{U_i \mid i \in I\}$ be an open covering of M by domains of charts. Let ρ_i be a subordinate partition of unity.

$$\omega = 1 \cdot \omega = \sum_{i\in I} \underbrace{\rho_i \omega}_{=:\omega_i} = \sum_{i \in I} \omega_i$$

where $\text{supp}(\omega_i) \subset U_i$. Define $d\omega = \sum_{i\in I} d\omega_i$, with $d\omega_i$ defined as follows: if

$$\omega_i = \varphi_i^* \left(\sum_{i_1<\cdots<i_k} f_{i_1,\dots,i_k}\ dx_{i_1} \wedge \cdots \wedge dx_{i_k}\right)$$

where it extended by 0 outside of U_i, then

$$d\omega_i = \varphi_i^* \left(\sum_{i_1<\cdots<i_k} df_{i_1,\dots,i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}\right)$$

where $df_{i_1,\dots,i_k}$ is defined by (1).

Well-definess: Suppose $\alpha \in \Omega^k(M)$, s.t. $\text{supp}(\alpha) \subset U_i \cap U_j$. Then $\varphi_i^*\beta = \alpha = \varphi_j^*\gamma$. We want to define $d\alpha$ as $\varphi_i^* d\beta$, so we need to check $\varphi_i^* d\beta = \varphi_j^* d\gamma$.

$$\begin{aligned}\gamma &= (\varphi_j^{-1})^*\alpha = (\varphi_j^{-1})^*\varphi_i^*\beta = (\varphi_i \circ \varphi_j^{-1})^*\beta\\ d\gamma &= d(\varphi_i \circ \varphi_j^{-1})^*\beta = (\varphi_i \circ \varphi_j^{-1})^* d\beta\end{aligned}$$

Therefore,

$$\varphi_i^* d\beta = \varphi_j^* d\gamma$$

□

Lemma 11.9. If $\alpha \in \Omega^1(M)$, then

$$d\alpha(X,Y) = L_X(\alpha(Y)) - L_Y(\alpha(X)) - \alpha([X,Y])$$

Proof. It is enough to check the formula for $\alpha = f \cdot dg$, where $f, g \in \mathcal{C}^\infty(M)$.

$$\begin{aligned}d\alpha &= df \wedge dg\\ d\alpha(X,Y) &= df \wedge dg(X,Y) = df(X)dg(Y) - df(Y)dg(X)\\ &= L_X f L_Y g - L_Y f L_X g\end{aligned}$$

$$
\begin{aligned}
&L_X(\alpha(Y)) - L_Y(\alpha(X)) - \alpha([X,Y]) \\
&= L_X(fdg(X)) - L_Y(fdg(X)) - fdg([X,Y]) \\
&= L_X(fL_Y(g)) - L_Y(fL_Xg) - fdg([X,Y]) \\
&= (L_Xf)(L_Yg) + fL_XL_Yg - (L_Yf)(L_Xg) - fL_YL_Xg - fdg([X,Y]) \\
&= (L_Xf)(L_Yg) - (L_Yf)(L_Xg) + \cancel{f(L_XL_Yg - L_YL_Xg - L_{[X,Y]}g)}
\end{aligned}
$$

□

Definition 11.4. For $X \in \mathfrak{X}(M)$, let φ_t be the flow. Then for $\omega \in \Omega^k(M)$, define the **Lie derivative** of ω as

$$L_X\omega = \frac{d}{dt}\varphi_t^*\omega\bigg|_{t=0}$$

Take $\alpha \in \Omega^1(M)$.

$$
\begin{aligned}
&(L_X\alpha)(Y)(p) = \left(\frac{d}{dt}\varphi_t^*\alpha\bigg|_{t=0}(p)\right)(Y(p)) \\
&= \lim_{t\to 0}\frac{\alpha(\varphi_t(p))(D_p\varphi_t(Y(p)) - \alpha(p)(Y(p)))}{t} \\
&= \lim_{t\to 0}\frac{\alpha(\varphi_t(p))(D_p\varphi_t(Y(p)) - Y(\varphi_t(p)) + \alpha(\varphi_t(p))(Y(\varphi_t(p))) - \alpha(p)(Y(p)))}{t} \\
&= \alpha(L_{-X}Y)(p) + L_X(\alpha(Y))(p) \\
&= L_X(\alpha(Y))(p) - \alpha([X,Y])(p)
\end{aligned}
$$

We have proved $(L_X\alpha)(Y) = L_X(\alpha(Y)) - \alpha([X,Y])$
$= \cancel{L_X(\alpha(Y))} + d\alpha(X,Y)\cancel{-L_X(\alpha(Y))} + L_Y(\alpha(X))$

$$\Rightarrow \qquad d\alpha(X,Y) = (L_X\alpha)(Y) - L_Y(\alpha(X))$$

Definition 11.5. For $X \in \mathfrak{X}(M)$, define the contraction with X

$$
\begin{aligned}
i_X : \Omega^k(M) &\to \Omega^{k-1}(M) \\
\omega &\mapsto \omega(X,\dots,X_k)
\end{aligned}
$$

by

$$i_X\omega(X_1,\dots,X_{k-1}) = \omega(X,X_1,\dots,X_k)$$

We have $i_X \equiv 0$ by skew-symmetry.

Moreover, $i_X(\omega\wedge\eta) = (i_X\omega)\wedge\eta + (-1)^{\deg\omega}\omega\wedge i_X\eta$. e.g. if $\deg\omega = \deg\eta = 1$, then

$$
\begin{aligned}
(i_X(\omega\wedge\eta))(Y) &= (\omega\wedge\eta)(X,Y) \\
&= \omega(X)\eta(Y) - \omega(Y)\eta(X) \\
&= ((i_X\omega)\wedge\eta)(Y) + (-1)^{\deg\omega}(\omega\wedge i_X\eta)(Y)
\end{aligned}
$$

In general, $\eta\wedge\omega = (-1)^{\deg\eta\cdot\deg\omega}\omega\wedge\eta$.

Theorem 11.10 (Cartan Formula)**.** On $\Omega^k(M)$, we have $L_X = d\circ i_X + i_X\circ d$.

Proof. For $k = 0$, the formula reduces to $L_X = i_X \circ d$. Apply L_X to $f \in \mathcal{C}^\infty(M)$: $L_X f = i_X df = df(X)$ true! For $k = 1$, let $\alpha \in \Omega^1(M)$, then

$$\begin{aligned}(L_X\alpha)(Y) &= d\alpha(X, Y) + L_Y(\alpha(X)) \\ &= d\alpha(X, Y) + d(\alpha(X))(Y) \\ &= ((i_X \circ d)\alpha)(Y) + ((d \circ i_X)\alpha)(Y)\end{aligned}$$

$$\Rightarrow \qquad L_X = i_X d + di_X \text{ on 1-forms}$$

In general, Cartan formula is local and $\mathbb{R}$-linear, so it is enough to check it on $\omega = \alpha_1 \wedge \cdots \wedge \alpha_k$, where $\alpha_i \in \Omega^1(M)$.

$$\begin{aligned}L_X\omega &= \sum_{j=1}^{k} \alpha_1 \wedge \cdots \wedge L_X\alpha_j \wedge \cdots \wedge \alpha_k \\ &= \sum_{j=1}^{k} (\alpha_1 \wedge \cdots \wedge i_X d\alpha_j \wedge \cdots \wedge \alpha_k + \alpha_1 \wedge \cdots \wedge d\alpha_i(X) \wedge \cdots \wedge \alpha_k) \\ &\overset{!}{=} i_X d\omega + d(i_X\omega)\end{aligned}$$

$$\begin{aligned}i_X\omega &= \sum_{j=1}^{k} (-1)^{i-1} \alpha_1 \wedge \cdots \wedge \alpha_j(X) \wedge \cdots \wedge \alpha_k \\ d\left(i_X\omega\right) &= \sum_{j=1}^{k} (-1)^{j-1} d\left(\alpha_j(X)\right) \wedge \alpha_1 \wedge \cdots \wedge \widehat{\alpha_j} \wedge \cdots \wedge \alpha_k \\ &+ \sum_{j=1}^{k} (-1)^{j-1} \alpha_j(X) d\left(\alpha_1 \wedge \cdots \wedge \widehat{\alpha_j} \wedge \cdots \wedge \alpha_k\right) \\ d\omega &= \sum_{j=1}^{k} (-1)^{j-1} \alpha_1 \wedge \cdots \wedge d\alpha_j \wedge \cdots \wedge \alpha_k. \\ i_X d\omega &= i_X \sum_{j=1}^{k} \alpha_1 \wedge \cdots \wedge d\alpha_j \wedge \cdots \wedge \alpha_k\end{aligned}$$

where the hat $(\widehat{\alpha_j})$ means that the jth factor is omitted. ☐

11.3 Manifolds with Boundary

We look at the half space

$$\mathbb{H}^n = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geqslant 0\} \subset \mathbb{R}^n$$

The boundary is

$$\partial\mathbb{H}^n = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n = 0\} = \mathbb{R}^{n-1} \subset \mathbb{H}^n$$

Then the interior is

$$\operatorname{int}\mathbb{H}^n = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\} \underset{\text{open}}{\subset} \mathbb{R}^n$$

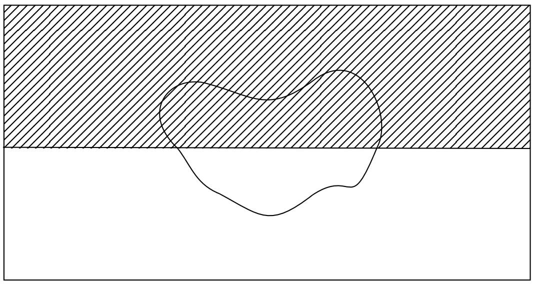
Definition 11.6. A differentiable manifold with boundary is a topological space M which is Hausdorff and has a countable basis for its topology and has an atlas (U_i, φ_i), $i \in I$, where $U_i \subset M$ are open, $M = \bigcup\limits_{i \in I} U_i$,

$$\varphi_i : U_i \to \mathbb{H}^n \text{ are homeomorphisms onto their images}$$

and, whenever $U_i \cap U_j \neq \varnothing$,

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j) \text{ is a diffeomorphism}$$

If $U \subset \mathbb{H}^n$ is open, a map $f : U \to N$ is differentiable if it admits a differentiable extension to open set in $\mathbb{R}^n$.



M manifold with boundary

$$\partial M := \{p \in M \mid \exists (U_i, \varphi_i), \text{ s.t. } \varphi_i(p) \in \partial\mathbb{H}^n\}$$
$$\operatorname{int} M := \{p \in M \mid \exists (U_i, \varphi_i), \text{ s.t. } \varphi_i(p) \in \operatorname{int}\mathbb{H}^n\}$$

Lemma 11.11. ∂M, $\operatorname{int} M$ are well-defined, $M = \partial M \cup \operatorname{int} M$.

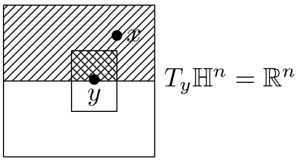
Proof. Suppose $p \in U_i$, and $\varphi_i(p) \subset \operatorname{int}\mathbb{H}^n$. If $p \in U_j$, then $\varphi_j \circ \varphi_i^{-1}$ maps $\varphi_i(U_i \cap U_j)$ diffeomorphically to $\varphi_j(U_i \cap U_j)$. If this touches $\partial\mathbb{H}^n$, shrink U_j, to get an open neighborhood of p in M. which maps to $\operatorname{int} M$. ☐

Considering $U_i \cap \operatorname{int} M$ and restricting φ_i, we obtain a smooth atlas for $\operatorname{int} M$, showing that $\operatorname{int} M$ is an n-dimensional manifold in the usual sense.

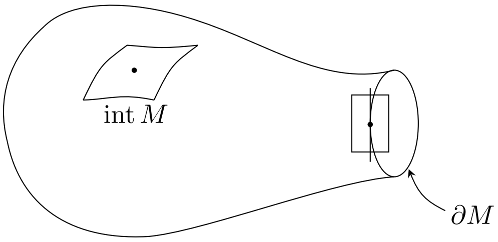
If $\partial M \neq \varnothing$, then considering $U_i \cap \partial M$ and restricting φ_i, we obtain a smooth atlas for ∂M, showing that ∂M is a $(n-1)$-dimensional smooth manifold in the usual sense.

$$\{\text{manifolds}\} \subset \{\text{manifolds with } \partial\}$$

$$(p, i, v),\ p \in M$$
$$i \in I, \text{ s.t. } p \in U_i$$
$$v \in \mathbb{R}^n = T_{\varphi_i(p)}\mathbb{R}^n$$



The usual definition of $TM \to W$ works for manifolds with boundary.



Example 11.1.

(0) $M = \mathbb{H}^n$, $\partial M = \partial \mathbb{H}^n$.

(1) $\overline{B_\varepsilon(p)} = \{x \in \mathbb{R}^n \mid d(x,p) \leqslant \varepsilon\}$, $\partial\overline{B_\varepsilon(p)} = S^{n-1}$.

(2) $$\left.\begin{array}{r} M \text{ manifold with boundary } \partial M \\ N\text{manifold (without boundary)} \end{array}\right\} \begin{array}{r} M \times N \text{ is a manifold with boundary} \\ \partial(M \times N) = \partial M \times N \end{array}$$

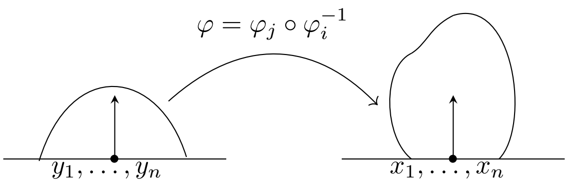
① M is orientable if $TM \to M$ is an orientable vector bundle.

② M is orientable if there is an atlas (U_i, φ_i), $i \in I$, s.t. all $\varphi_j \circ \varphi_i^{-1}$ are orientable-preserving.

①⇔②: Both definitions also apply to manifolds with boundary, and are still equivalent.

Lemma 11.12. If M is an orientable manifold with boundary, then ∂M is an orientable manifold (in the usual sense).

Proof. M is orientable $\Rightarrow \exists$ atlas, s.t. all $\varphi_j \circ \varphi_i^{-1}$ are orientable-preserving. Suppose $p \in \partial M$, and $p \in U_i \cap U_j$.



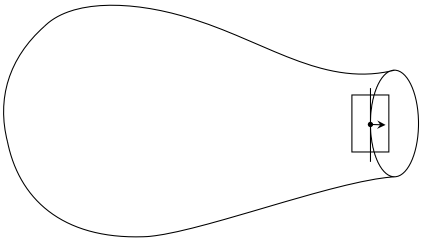
$x_k = \varphi_k(y_1, \dots, y_n)$

$$D_{\varphi_i(p)}(\varphi) = \left(\frac{\partial \varphi_k}{\partial y_l}\right)$$
$$= \begin{pmatrix} & & * \\ \frac{\partial \varphi_k}{\partial y_l}, k, l \leqslant n-1 & & \vdots \\ & & \vdots \\ & & * \\ 0 \ \cdots \ \cdots \ 0 & & \frac{\partial \varphi_n}{\partial y_n} \end{pmatrix}$$

where $\frac{\partial \varphi_n}{\partial y_n} > 0$. Since $D_{\varphi_i(p)}\varphi$ is orientation-preserving, the restriction $\varphi\Big|_{\partial \mathbb{H}}$ is also orientation-preserving. □

Let $x_1, \dots, x_n$ be the linear coordinates on $\mathbb{R}^n$, $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ is an oriented basis for $\mathbb{R}^n = T_0\mathbb{R}^n$. We want to choose a basis $v_1, \dots, v_{n-1}$ for $\mathbb{R}^{n-1} \subset \mathbb{R}^n$, so that $-\frac{\partial}{\partial x_n}$, $v_1, \dots, v_{n-1}$ is positively oriented in $\mathbb{R}^n$, i.e. it defines the same orientation as $\dfrac{\partial}{\partial x_1}, \dots, \dfrac{\partial}{\partial x_n}$.

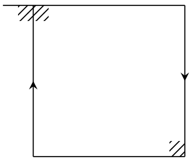
$$v_1 = (-1)^n \frac{\partial}{\partial x_1}$$
$$v_2 = \frac{\partial}{\partial x_2}$$
$$\vdots$$
$$v_{n-1} = \frac{\partial}{\partial x_{n-1}}$$



Definition 11.7. If M is oriented, so that $\dfrac{\partial}{\partial x_1}, \dots, \dfrac{\partial}{\partial x_n}$ give the orientation in a chart, then $v_1, \dots, v_{n-1}$ define an orientation for ∂M, called the **induced orientation on the boundary**.

Example 11.2. $M := ([0,1] \times [0,1])/\sim$ is a manifold with boundary $\partial M \cong S^1$.

$$(0,t) \sim (1, 1-t)$$



Remark. If M is orientable, so is $\operatorname{int} M$.

11.4 Stokes' Theorem

$\omega \in \Omega^n(\mathbb{R}^n) \Rightarrow \omega = f \cdot dx_1 \wedge \dots \wedge dx_n$, $f \in \Omega^0(\mathbb{R}^n)$. ω has compact support $\Leftrightarrow$ f has compact support.

Assume this is the case. Define

$$\int \omega = \int f dx_1 \cdots dx_n$$

Let φ be an orientation-preserving diffeomorphism.

$$\int \varphi^* \omega = \int (f \circ \varphi) \det \left(\frac{\partial \varphi_i}{\partial x_j} \right) dx_1 \cdots dx_j$$

We get $\int\limits_{\mathbb{R}^n} \varphi_* \omega = \int\limits_{\mathbb{R}^n} \omega$ if $\det \left(\dfrac{\partial \varphi_i}{\partial x_j} \right) > 0$. So $\int$ is well-defined under orientation-preserving changes of coordinates. Let M be an orientable manifold with fixed orientation. Let (U_i, φ_i) be a smooth orientated atlas.

Theorem 11.13. There is a well-defined $\mathbb{R}$-linear map (works for M with boundary by $\mathbb{R}^n \mapsto \mathbb{H}^n$)

$$\int\limits_M : \Omega_0^n(M) \to \mathbb{R}$$

s.t. if $\operatorname{supp}(\omega) \subset U_i$, then $\displaystyle\int\limits_M \omega = \int\limits_{\mathbb{R}^n} (\varphi_i^{-1})^* \omega$.

Proof. Let $\omega \in \Omega_0^n(M)$, and let ρ_i be a smooth partition of unity subordinate to U_i.

$$\omega = 1 \cdot \omega = \sum_{i \in I} (\rho_i \omega)$$

If $\int\limits_M$ exists and is $\mathbb{R}$-linear, then

$$\int\limits_M \omega = \sum_i \int\limits_M (\rho_i \omega) = \sum_i \int\limits_{\mathbb{R}^n} (\varphi_i^{-1})^* (\rho_i \omega) \tag{11.1}$$

This shows that $\int\limits_M$ is unique. Use (11.1) to define $\int\limits_M$. This is well-defined, because all transition maps are orientation preserving, so

$$\int\limits_{\mathbb{R}^n} (\varphi_i^{-1})^* (\omega_i) = \int\limits_{\mathbb{R}^n} (\varphi_j^{-1})^* (\omega_i)$$

if $\operatorname{supp}(\omega_i) \subset U_i \cap U_j$. ☐

Let M be a smooth n-dimensional manifold with boundary.

Theorem 11.14 (Stoke's Theorem)**.** If $i : \partial M \hookrightarrow M$ is the inclusion and M is orientable, then

$$\int\limits_M d\omega = \int\limits_{\partial M} i^* \omega \qquad \forall \omega \in \Omega_0^{n-1}(M)$$

where ∂M carries the orientation induced from M.

Proof. Case 1: $M = \mathbb{H}^n$.

$$\omega = \sum_{i=1}^n f_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

where the hat $(\widehat{dx_i})$ means that the ith factor is omitted. Since i^*, d, $\int$ are $\mathbb{R}$-linear, we prove stokes theorem for each summand. Without loss of generality, $\omega = f dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$.

Subcase 1a: $i < n \Rightarrow i^*\omega \equiv 0 \Rightarrow \int\limits_{\partial \mathbb{H}^n} i^*\omega = 0$.

$$\begin{aligned}
\int\limits_{\mathbb{H}^n} d\omega &= \int\limits_{\mathbb{H}^n} \sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \\
&= \int\limits_{\mathbb{H}^n} \frac{\partial f}{\partial x_i} dx_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \\
&= (-1)^{i-1} \int\limits_{\mathbb{H}^n} \frac{\partial f}{\partial x_i} dx_1 \wedge \cdots \wedge dx_n \\
&= (-1)^{i-1} \int \cdots \int \frac{\partial f}{\partial x_j} dx_1 \cdots dx_n \\
&= (-1)^{i-1} \int \cdots \int \int_{-\infty}^{+\infty} \frac{\partial f}{\partial x_i} dx_i dx_1 \cdots \widehat{dx_i} \cdots dx_n = 0
\end{aligned}$$

$\int_{-\infty}^{+\infty} \frac{\partial f}{\partial x_i} dx_i = 0$, since f has compact support.

Subcase 1b: $i = n$, $\omega = f dx_1 \wedge \cdots \wedge dx_{n-1}$. $i^*\omega = f\Big|_{\partial \mathbb{H}^n} dx_1 \wedge \cdots \wedge dx_{n-1}$.

$$\int\limits_{\partial \mathbb{H}^n} i^*\omega = (-1)^n \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f\Big|_{\mathbb{H}^n} dx_1 \cdots dx_{n-1}$$

where the equality comes from induced orientation, since $(-1)^n dx_1 \cdots dx_{n-1}$ is positive oriented. Then

$$\begin{aligned}\int\limits_{\mathbb{H}^n} d\omega &= \int\limits_{\mathbb{H}^n} \frac{\partial f}{\partial x_n} dx_n \wedge dx_1 \wedge \cdots \wedge dx_{n-1} \\ &= (-1)^{n-1} \int \frac{\partial f}{\partial x_n} dx_1 \wedge \cdots \wedge dx_n \\ &= (-1)^{n-1} \int\limits_{\mathbb{H}^n} \frac{\partial f}{\partial x_n} dx_1 \cdots dx_n \\ &= (-1)^{n-1} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \int_0^{+\infty} \frac{\partial f}{\partial x_n} dx_n dx_1 \cdots dx_{n-1} \\ &= (-1)^n \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f\Big|_{\partial \mathbb{H}^n} dx_1 \cdots dx_{n-1}\end{aligned}$$

where $\int_0^{+\infty} \frac{\partial f}{\partial x_n} dx_n = \underbrace{f(x_1, \ldots, x_{n-1}, \infty)}_{=0} - f(x_1, \ldots, x_{n-1}, 0) = -f\Big|_{\mathbb{H}^n}$.

Case 2: M arbitrary, $\omega \in \Omega_0^{n-1}(M)$, $\omega = \sum\limits_i \rho_i \omega$.

$$\begin{aligned}\int\limits_M d\omega &= \sum_i \int\limits_M d(\rho_i \omega) \\ &= \sum_i \int\limits_{\mathbb{H}^n} (\varphi_i^{-1})^* d(\rho_i \omega) \\ &= \sum_i \int\limits_{\mathbb{H}^n} d(\varphi_i^{-1*}(\rho_i \omega)) \\ &\overset{\text{Case 1}}{=\!=\!=} \sum_i \int\limits_{\partial \mathbb{H}^n} i^*(\varphi_i^{-1})^*(\rho_i \omega) \\ &= \sum_i \int\limits_{\partial \mathbb{H}^n} (\varphi_i^{-1})^* i^*(\rho_i \omega) = \int\limits_{\partial M} i^*\omega\end{aligned}$$

□

Corollary 11.15. If $\partial M = \varnothing$, then $\int\limits_M d\omega = 0$.

Corollary 11.16. If $\partial M = \varnothing$, then M does not have a volume form ω, which is $d\alpha$, with $\alpha \in \Omega_0^{n-1}(M)$.

Proof.

$$0 < \int\limits_M \omega = \int\limits_M d\alpha = 0$$

This leads to a contradiction. □

Example 11.3. $M = B_1(0) \subset \mathbb{R}^2$, $\partial M = S^1$, $\omega = dx \wedge dy = d(xdy)$. Then

$$\text{Area}(B_1(0)) = \int\limits_M d(xdy) = \int\limits_{S^1} xdy = \int\limits_{S^1} \cos\varphi d(\sin\varphi) = \int\limits_{S^1} \cos^2\varphi d\varphi$$

Since

$$(\sin\varphi\cos\varphi)' = \cos^2\varphi - \sin^2\varphi = 2\cos^2\varphi - 1 \;\Rightarrow\; \cos^2\varphi = \frac{1}{2} + \frac{1}{2}(\sin\varphi\cos\varphi)'$$

Thus,

$$\text{Area}(B_1(0)) = \int\limits_{S^1} \frac{1}{2} + \frac{1}{2}\int\limits_{S^1} (\sin\varphi\cos\varphi)' d\varphi = \int_0^{2\pi} \frac{1}{2} d\varphi = \pi$$

Chapter 12

de Rham Cohomology

$$H^*_{dR}(M) = \bigoplus_{k=0}^{n} H^k_{dR}(M)$$

Definition 12.1. $H^k_c(M) := \dfrac{\ker(d : \Omega^k_0(M) \to \Omega^{k+1}_0(M))}{\operatorname{im}(d : \Omega^{k-1}_0(M) \to \Omega^k_0(M))}$ the de Rham cohomology of M with compact support, where $\ker(d : \Omega^k_0(M) \to \Omega^{k+1}_0(M))$ is the closed form and $\operatorname{im}(d : \Omega^{k-1}_0(M) \to \Omega^k_0(M))$ is the exact form.

Example 12.1. $H^0_c(M)$ =locally constant functions with compact support. If M is connected, then

$$H^0_c(M) = \begin{cases} \mathbb{R} & M \text{ compact} \\ 0 & M \text{ non-compact} \end{cases}$$

$M = \mathbb{R}$, $k = 1$

$$H^1_c(\mathbb{R}) = \frac{\Omega^1_c(\mathbb{R})}{\operatorname{im}(d : \Omega^0_c(\mathbb{R}) \to \Omega^1_c(\mathbb{R}))}$$

$\Omega^1_c(\mathbb{R}) \ni \omega = f dt$, $f \in \mathcal{C}^\infty_0(\mathbb{R})$. Before

$$F(x) = \int_{-\infty}^{x} f(t) dt, \quad \omega = dF$$

But F does not have compact support, if $\displaystyle\int\limits_{-\infty}^{+\infty} f(t) dt = c \neq 0$.

If $c = 0$, then $F \in \Omega^0_0(\mathbb{R})$ and $dF = \omega$, then $[\omega] = 0 \in H^1_c(\mathbb{R})$. If $c \neq 0$, then still $dF = \omega$, but $F \notin \Omega^0_0(\mathbb{R})$.

Suppose $G \in \mathcal{C}^\infty_0(\mathbb{R})$ and $dG = \omega$. Then $d(F - G) = 0$. $\Rightarrow F - G = d$ is constant, for $x \ll 0$: $G(x) = d$ and for $x \gg 0$: $G(x) = -d + c$.

Since G has compact support $\Rightarrow d = 0$ and $d = c$. This leads to a contradiction since $c \neq 0$.

$\Rightarrow \omega \notin \operatorname{im}(d : \Omega^0_0(\mathbb{R}) \to \Omega^1_0(\mathbb{R}))$.

$\Rightarrow H^1_c(\mathbb{R}) \neq 0$.

Theorem 12.1. If M is a smooth n-dimensional oriented manifold without boundary, then

$$\int_M : H_c^n(M) \to \mathbb{R}$$

is well-defined and surjective.

Proof. If $[\omega'] = [\omega] \in H_c^n(M)$, then $\omega' = \omega + d\alpha$, with $\alpha \in \Omega_0^{n-1}(M)$.

$$\int_M \omega' = \int_M \omega + \int_M d\alpha \overset{\text{Stokes}}{=\!=\!=} \int_M \omega + \int_{\partial M} \alpha = \int_M \omega$$

Let $\rho \geqslant 0$ be a bump function, with support in a chart: $\omega = \rho dx_1 \wedge \cdots \wedge dx_n$.

$$\int_M \omega = \int_U \rho dx_1 \wedge \cdots \wedge dx_n = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \rho dx_1 \cdots dx_n > 0$$

By linearity, surjective follows. ☐

Example 12.2. $M = \mathbb{R}$

$$\int_{\mathbb{R}} : H_c^1(\mathbb{R}) \to \mathbb{R}$$

Claim 12.2. This is an isomorphism.

Proof. $fdt = \alpha \in \Omega_0^1(\mathbb{R})$, where $f \in \Omega_0^0(\mathbb{R})$.

$$\int_{\mathbb{R}} \alpha = \int_{\mathbb{R}} fdt = c$$

If $[\alpha] \in \ker \left(\int_{\mathbb{R}} \right)$

$$\begin{aligned} \Leftrightarrow \quad & c = 0 \\ \Leftrightarrow \quad & \alpha = dF \text{ with } F \in \Omega_0^0(\mathbb{R}) \\ \Leftrightarrow \quad & [\alpha] = 0 \end{aligned}$$

The instance of Poincaré duality.

$M = \mathbb{R}$	$H_{dR}^k(\mathbb{R})$	$H_c^k(\mathbb{R})$
$k = 0$	$\mathbb{R}$	0
$k = 1$	0	$\mathbb{R}$

☐

$H_c^*(M) = \bigoplus_{k=0}^{n} H_c^k(M)$ is an algebra with $\wedge$ induced by wedge product on forms.

$$H_{dR}^k(M) \times H_c^l(M) \to H_c^{k+l}(M)$$

because a wedge product has compact support if one of the factors does.

Suppose $f : M \to N$ is a smooth map between smooth manifolds. The pullback $f^* : \Omega^k(N) \to \Omega^k(M)$ commutes with d. In particular, if $d\omega = 0$, then $df^*\omega = f^*d\omega = 0$ and if $\omega = d\alpha$, $f^*\omega = df^*\alpha$.

$\Rightarrow \begin{array}{c} f^* : H^k_{dR}(N) \to H^k_{dR}(M) \\ [\omega] \mapsto [f^*\omega] \end{array}$ is well-defined, linear. This f^* induces an algebra homomorphism.

$$f^* : H_{dR}(N) \to H_{dR}(M)$$

Example 12.3. $M = \overline{B_1(0)} \subset \mathbb{R}^n$.

Claim 12.3. There is no smooth map $n : M \to \partial M$, $r\Big|_{\partial M} = \mathrm{Id}\Big|_{\partial M}$.

Proof. Assume there is such an r, then

$$\begin{tikzcd}
\partial M = S^{n-1} \arrow[r, "i"] \arrow[rr, bend right, "\mathrm{Id}_{S^{n-1}}"'] & \overline{B_1(0)} = M \arrow[r, "r"] & \partial M = S^{n-1}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{R} \arrow[d, equal] & 0 \arrow[d, equal] & \mathbb{R} \arrow[d, equal] \\
H^{n-1}_{dR}(S^{n-1}) & H^{n-1}_{dR}(\overline{B_1(0)}) \arrow[l, "i^*"'] & H^{n-1}_{dR}(S^{n-1}) \arrow[l, "r^*"'] \arrow[ll, bend left, "=(r\circ i)^*=\mathrm{Id}^*_{S^{n-1}}=\mathrm{Id}"]
\end{tikzcd}$$

This leads to a contradiction. □

Chapter 13

Connections and Curvature

13.1 Connection

Let $E \to M$ be a smooth vector bundle of rank k. Then

$$\Gamma(T^*M \otimes E) = \Omega^1(E)$$

which is adjunction $T^*M \otimes E \leftrightarrow \operatorname{Hom}(TM, E)$. If $\alpha \in \Omega^1(E)$, then $\alpha(X) \in \Gamma(E)$, $\forall X \in \mathfrak{X}(M)$.

Definition 13.1. A **connection** ∇ on E is a $\mathbb{R}$-linear map

$$\nabla : \Gamma(E) \to \Omega^1(E)$$

satisfying the Leibniz rule

$$\nabla(f \cdot s) = df \otimes s + f\nabla(s), \qquad \forall f \in \mathcal{C}^\infty(M), s \in \Gamma(E)$$

Properties:

(1) ∇ does not increase the support of a section, i.e. if $U \subset M$ open and $s \in \Gamma(E)$, $s\Big|_U \equiv 0$, then $\nabla s\Big|_U \equiv 0$.

Proof. Take $p \in U$. Then there exists a smooth function $f \in \mathcal{C}^\infty(M)$, with $f(p) = 1$ and $\operatorname{supp} f \subset U$. Then $f \cdot s \equiv 0$, so by $\mathbb{R}$-linearity:

$$0 = \nabla(f \cdot s) = df \otimes s + f\nabla(s)$$

Evaluated at p

$$0 = \underbrace{(df \otimes s)(p)}_{=0,\text{ because } s(p)=0} + f(p)\nabla(s)(p) = \nabla(s)(p)$$

This implies $\nabla(s) = 0$ on U, because $p \in U$ was arbitrary. □

(2) The value of $\nabla(s)$ at any point $p \in M$ depends only on the restriction of s to an arbitrarily small neighborhood of p. If $s, s' \in \Gamma(E)$, s.t. $s\Big|_U \equiv s'\Big|_U$ for some $U \ni p$, then
$$\nabla(s)\Big|_U = \nabla(s')\Big|_U$$
$\nabla(s)(p)$ depends only on the germ of s at p. This is called differential operator.

(3) If ∇^1 and ∇^0 are connections, so is $t\nabla^1 + (1-t)\nabla^0 := \nabla^t$, $\forall t \in \mathbb{R}$.

Proof. ∇^t is $\mathbb{R}$-linear.
$$\begin{aligned}\nabla^t(fs) &= t\nabla^1(fs) + (1-t)\nabla^0(fs)\\ &= t(df \otimes s + f\nabla^1(s)) + (1-t)(df \otimes s + f\nabla^0(s))\\ &= df \otimes s + f\nabla^t(s)\end{aligned}$$
☐

(4) If ∇^1, ∇^0 are connections, then $\nabla^1 - \nabla^0 \in \Omega^1(\mathrm{End}(E))$, $\mathrm{End}(E) = \mathrm{Hom}(E, E) = E^* \otimes E$.

Proof. $\nabla^1 - \nabla^0$ is $\mathbb{R}$-linear.
The Leibniz rule gives $(\nabla^1 - \nabla^0)(fs) = f(\nabla^1 - \nabla^0)(s)$.
$\Rightarrow$ $(\nabla^1 - \nabla^0)(s)(p)$ depends only on $s(p)$.
$(\nabla^1 - \nabla^0)_p : E_p \to T_p^*M \otimes E_p$.
$\nabla^1 - \nabla^0 \in \Gamma(E^* \otimes T^*M \otimes E) = \Omega^1(E^* \otimes E) = \Omega^1(\mathrm{End}(E))$. ☐

Proposition 13.1. Every vector bundle E admits connections. The space of connections is naturally an affine space whose vector space of translation is $\Omega^1(\mathrm{End}(E))$.

Proof. Suppose E has connections. Then the difference of two connections is an element in $\Omega^1(\mathrm{End}(E))$ by (4). Conversely, let $A \in \Omega^1(\mathrm{End}(E))$ and ∇ a connection on E. ☐

Claim 13.2. $\begin{aligned}\nabla + A : \Gamma(E) &\to \Omega(E)\\ s &\mapsto \nabla(s) + A(s)\end{aligned}$ is a connection.

Proof. $\nabla + A$ $\mathbb{R}$-linear is clear.
$(\nabla + A)(fs) = \nabla(fs) + A(fs) = df \otimes s + f\nabla(s) + fA(s) = df \otimes s + f(\nabla + A)(s)$. ☐

13.2 Existence of Connections

Pick a system of local trivializations for E.

$$\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^k$$

On $E\Big|_{U_i}$, we define ∇^i as follows: Let $s_j \in \Gamma(E\big|_{U_i})$ be defined by $s_j(p) = \psi_i^{-1}(p, e_j)$, where $e_1, \dots, e_k$ is the standard basis of $\mathbb{R}^k$. Every section $s \in \Gamma(E\big|_{U_i})$ has the form $s = \sum\limits_{j=1}^{k} f_j s_j$ for uniquely determined functions $f_j \in \mathcal{C}^\infty(U_i)$.

$$\nabla^i(s) := \sum_{j=1}^{k} df_j \otimes s_j$$

This is clearly $\mathbb{R}$-linear.

$$\begin{aligned}\nabla^i(fs) &= \nabla^i\left(\sum_{j=1}^{k} f \cdot f_j s_j\right)\\ &= \sum_{j=1}^{k} d(ff_j) \otimes s_j\\ &= \sum_{j=1}^{k} (f df_j + f_j df) \otimes s_j\\ &= df \otimes s + f \cdot \nabla^i(s)\end{aligned}$$

so ∇^i is a connection on $E\Big|_{U_i}$.

Let ρ_i be a smooth partition of unity subordinate to the covering of M by the U_i. Define $\nabla := \sum\limits_i \rho_i \nabla^i \cdot As$ in (3), we can show that ∇ is a connection. $\nabla^i(s_j) = 0$ by definition.

Terminology. $s_1, \dots, s_k$ form a frame for $E\Big|_{U_i}$.

If s is a section, s.t. $\nabla(s) \equiv 0$ for some connection ∇, then s is called **parallel** or **covariantly constant**.

$\Omega^l(E) = \Gamma(\Lambda^l T^* M \otimes)$ l-forms on M, with values in E.

Lemma 13.3. For every connection ∇ on E, there is a unique $\mathbb{R}$-linear map $\overline{\nabla} : \Omega^l(E) \to \Omega^{l+1}(E)$ satisfying:

$$\overline{\nabla}(\omega \otimes s) = d\omega \otimes s + (-1)^l \omega \wedge \nabla s, \qquad \forall \omega \in \Omega^l(M), s \in \Gamma(E) \tag{13.1}$$

Moreover, this $\overline{\nabla}$ satisfies

$$\overline{\nabla}(f(\omega \otimes s)) = (df \wedge \omega) \otimes s + f\overline{\nabla}(\omega \otimes s), \qquad f \in \mathcal{C}^\infty(M)$$

Proof. Every element in $\Omega^l(E)$ is locally a sum of terms of the form $\omega \otimes s$. Define $\overline{\nabla}$ using a partition of unity and linearity, so $\overline{\nabla}$ is uniquely determined by (13.1).

$$\begin{aligned}\overline{\nabla}(f(\omega \otimes s)) &= \overline{\nabla}((f\omega) \otimes s)\\ &= d(f\omega) \otimes s + (-1)^l f(\omega \wedge \nabla s)\\ &= (df \wedge \omega) \otimes s + f d\omega \otimes s + f(-1)^l \omega \wedge \nabla s\\ &= (df \wedge \omega) \otimes s + f\nabla(\omega \otimes s)\end{aligned}$$

□

13.3 Curvature

Let ∇ be a connection on E.

Lemma 13.4. The composition $\overline{\nabla} \circ \nabla : \Gamma(E) \to \Omega^2(E)$ is function-linear.

Proof.

$$\begin{aligned}(\overline{\nabla} \circ \nabla)(fs) &= \overline{\nabla}(df \otimes s + f\nabla(s))\\ &= \overline{\nabla}(df \otimes s) + \overline{\nabla}(f\nabla(s))\\ &= \underbrace{dd}_{d^2\equiv 0} f \otimes s - df \wedge \nabla s + df \wedge \nabla s + f\overline{\nabla}(\nabla s)\\ &= f(\overline{\nabla} \circ \nabla)(s)\end{aligned}$$

□

The lemma shows that $\overline{\nabla}\nabla(s) = F^{\nabla}(s)$, where $F^{\nabla} \in \Omega^2(\mathrm{End}(E))$.

Definition 13.2. F^{∇} is called the **curvature** of ∇.

Let $E \to M$ be smooth vector bundle with connection ∇. Let $s_1, \ldots, s_k$ be frame. Then

$$\nabla(s_i) = \sum_{j=1}^{k} \omega_{ij} \otimes s_j$$

where $\omega_{ij} \in \Omega^1(M)$. We have $s = \sum_{i=1}^{k} f_i s_i$, $\nabla(s) = \sum_{i=1}^{k}(df_i \otimes s_i + f_i \nabla(s_i))$ completely determined by (ω_{ij}). $F^{\nabla} \in \Omega^2(\mathrm{End}(E))$.

$$F^{\nabla}(s_i) = \sum_{j=1}^{k} \Omega_{ij} \otimes s_j$$

where $\Omega_{ij} \in \Omega^2(M)$.

Question: How is Ω_{ij} determined by ω_{ij}?

$$\begin{aligned}
F^{\nabla}(s_i) &= \overline{\nabla} \circ \nabla(s_i) \\
&= \overline{\nabla}\left(\sum_j \omega_{ij} \otimes s_j\right) \\
&= \sum_{j=1}^{k} (d\omega_{ij} \otimes s_j - \omega_{ij} \wedge \nabla(s_j)) \\
&= \sum_{j=1}^{k} \left[d\omega_{ij} \otimes s_j - \omega_{ij} \wedge \sum_{l=1}^{k} \omega_{lj} \otimes s_l\right] \\
&= \sum_{j=1}^{k} \left[d\omega_{ij} - \sum_{l=1}^{k} \omega_{il} \wedge \omega_{lj}\right] \otimes s_j.
\end{aligned}$$

So

$$\Omega_{ij} = d\omega_{ij} - \sum_{l=1}^{k} \omega_{il} \wedge \omega_{lj}$$

This can be denoted as

$$\boxed{\Omega = d\omega - \omega \wedge \omega}$$

Then

$$\begin{aligned}
d\Omega_{ij} &= -\sum_l d\omega_{il} \wedge \omega_{lj} + \sum_l \omega_{il} \wedge d\omega_{lj} \\
&= -\sum_l \left[\Omega_{il} + \sum_m \omega_{im} \wedge \omega_{ml}\right] \wedge \omega_{lj} + \sum_l \left[\omega_{il} \wedge \left(\Omega_{lj} + \sum_m \omega_{lm} \wedge \omega_{mj}\right)\right] \\
&= \sum_l (\omega_{il} \wedge \Omega_{lj} - \Omega_{il} \wedge \omega_{lj}) + \underbrace{\sum_{l,m} (\omega_{il} \wedge \omega_{lm} \wedge \omega_{mj} - \omega_{im} \wedge \omega_{ml} \wedge \omega_{lj})}_{=0}
\end{aligned}$$

13.4 Bianchi Identity

$$d\Omega_{ij} = \sum_l \left[\omega_{il} \wedge \Omega_{lj} - \Omega_{il} \wedge \omega_{lj}\right]$$

which can be denoted as

$$\boxed{d\Omega = \omega \wedge \Omega - \Omega \wedge \omega}$$

Let $s'_1, \dots, s'_k$ be another frame.

$$s'_i = \sum g_{ij} s_j$$

where $g = (g_{ij})$ invertible. Then

$$\nabla(s'_i) = \sum_{j=1}^{k} \omega'_{ij} s'_j$$

where $(\omega_{ij})'$ is the connection matrix of ∇ with respect to $s_1', \ldots, s_k'$.

$$\begin{aligned}
\nabla(s_i') &= \nabla \left(\sum_j g_{ij} s_j \right) \\
&= \sum_j (dg_{ij} \otimes s_j + g_{ij} \nabla(s_j)) \\
&= \sum_j \left(dg_{ij} \otimes s_j + g_{ij} \sum_{l=1}^{k} \omega_{jk} \otimes s_l \right) \\
&= \sum_{l=1}^{k} \left(dg_{ij} + \sum_{j=1}^{k} g_{ij} \omega_{jl} \right) \otimes s_l \\
&= \sum_{l=1}^{k} \left(dg_{il} + \sum_{j} g_{ij} \omega_{jl} \right) \otimes \sum_{m=1}^{k} g_{lm}^{-1} s_m' \\
&= \sum_{m=1}^{k} \left(\left(\sum_l dg_{il} + \sum_j g_{ij} \omega_{jl} \right) g_{lm}^{-1} \right) \otimes s_m'
\end{aligned}$$

Then

$$\omega_{im}' = \sum_l \left(dg_{il} + \sum_j g_{ij} \omega_{jl} \right) g_{lm}^{-1}$$

This can be denoted as

$$\boxed{\omega' = dgg^{-1} + g\omega g^{-1}}$$

The following terms are the same:

$$\begin{aligned}
\text{a choice of local trivialization} &\Leftrightarrow \text{a choice of a frame} \\
&\Leftrightarrow \text{a choice of gauge}
\end{aligned}$$

A change of frame is called a **gauge transformation** g.

$$\begin{aligned}
\Omega' &= d\omega' - \omega' \wedge \omega' \\
&= d(dgg^{-1} + g\omega g^{-1}) - (dgg^{-1} + g\omega g^{-1}) \wedge (dgg^{-1} + g\omega g^{-1}) \\
&= d^2 gg^{-1} - dg \wedge dg^{-1} + dg \wedge \omega g^{-1} + gd\omega g^{-1} - g\omega \wedge dg^{-1} \\
&- dgg^{-1} \wedge dgg^{-1} - g\omega g^{-1} \wedge dgg^{-1} - dgg^{-1} \wedge g\omega g^{-1} - g\omega g^{-1} \wedge g\omega g^{-1} \\
&= g\Omega g^{-1}
\end{aligned}$$

Compare the following two equation:

$$\begin{aligned}
\omega' &= dgg^{-1} + g\omega g^{-1} \\
\Omega' &= g\Omega g^{-1}
\end{aligned}$$

The first one shows that connection is not a "tensor", while the second one shows that curvature is a "tensor".

Let $\nabla : \Gamma(E) \to \Omega(E)$.

Definition 13.3. $\nabla_X s = \langle \nabla s, X \rangle \in \Gamma(E)$ for every $X \in \mathfrak{X}(M)$ is the covariant derivative of s (in the direction of X).

Let (U, φ) be a chart for M.

$$\begin{aligned} x_1, \ldots, x_n : U &\to \mathbb{R} \\ p &\mapsto \varphi_i(p) \end{aligned}$$

where $dx_1, \ldots, dx_n$ are the dual frame to $\mathcal{O}_1, \ldots, \mathcal{O}_n$.

Let $s_1, \ldots, s_k$ be a frame for $E\Big|_U$. ∇ is represented by $\omega = (\omega_{ij})$ with respect to $s_1, \ldots, s_k$. Then $\omega_{ij} = \sum\limits_\alpha^n {\omega_{ij}}^\alpha dx_\alpha$ for unique ${\omega_{ij}}^\alpha \in \mathcal{C}^\infty(U)$, where ${\omega_{ij}}^\alpha = \langle \omega_{ij}, \partial_\alpha \rangle$.

$$\nabla_{\partial_\alpha} s_i = \langle \nabla s_i, \partial_\alpha \rangle = \left\langle \sum_j \omega_{ij} \otimes s_j, \partial_\alpha \right\rangle = \sum_j \langle \omega_{ij}, \partial_\alpha \rangle \otimes s_j = \sum_j {\omega_{ij}}^\alpha s_j$$

$$s = \sum_{i=1}^k f_i s_i$$

$$\begin{aligned} \nabla_{\partial_\alpha} s = \langle \nabla s, \partial_\alpha \rangle &= \left\langle \sum_{i=1}^k df_i \otimes s_i + f_i \nabla s_i, \partial_\alpha \right\rangle = \sum_{j=1}^k \partial_\alpha f_j s_j + \sum_{i,j} f_i {\omega_{ij}}^\alpha s_j \\ &= \sum_{j=1}^k \left(\partial_\alpha f_j + \sum_{i=1}^k f_i {\omega_{ij}}^\alpha \right) s_j \end{aligned}$$

$A^\alpha := ({\omega_{ij}}^\alpha)$ is $k \times k$ matrix of $\mathcal{C}^\infty$-functions on U.

$$\nabla_{\partial_\alpha} s = \partial_\alpha s + A^\alpha(s)$$

We define

$$\nabla_\alpha := \partial_\alpha + A^\alpha$$

13.5 Parallel Transport

Let $M \subset \mathbb{R}^n$ open. Let $E \xrightarrow{\pi} M$ smooth vector bundle of rank k, with a connection ∇. Let $y_1, \ldots, y_n$ be linear coordinates on $\mathbb{R}^n$. Let $c : [0,1] \to M$ be smooth curve, then write it is in terms of coordinates

$$\begin{aligned} c(t) &= (y_1(t), \ldots, y_n(t)) \\ \dot{c}(t) &= \sum_{\alpha=1}^n \frac{dy_\alpha}{dt} \frac{\partial}{\partial y_\alpha} = \sum_{\alpha=1}^n \dot{y}_\alpha \partial_\alpha \end{aligned}$$

Assume E is trivial. $E \cong M \times \mathbb{R}^k$. Let $s_1, \dots, s_k$ be the frame with $s_i(p) = (p, e_i)$. $s \in \Gamma(E)$ is written uniquely as $s = \sum_{i=1}^{k} x_i s_i$.

$$\begin{aligned}
\nabla_{\dot{c}}(s) &= \langle \nabla s, \dot{c} \rangle \\
&= \left\langle \sum_{i=1}^{k} dx_i \otimes s_i + x_i \nabla s_i, \sum_{\alpha=1}^{n} \dot{y}_\alpha \partial_\alpha \right\rangle \\
&= \sum_{i,\alpha} \left\langle \sum_{j=1}^{n} \frac{\partial x_i}{\partial y_j} dy_j \otimes s_i + x_i \sum_{j=1}^{k} \omega_{ij} \otimes s_j, \dot{y}_\alpha \partial_\alpha \right\rangle \\
&= \sum_{i,\alpha} \dot{y}_\alpha \frac{\partial x_i}{\partial y_\alpha} s_i + x_i \sum_{j=1}^{n} {\omega_{ij}}^\alpha \dot{y}_\alpha s_j \\
&= \sum_{j,\alpha} \dot{y}_\alpha \frac{\partial x_j}{\partial y_\alpha} s_j + \sum_{i,j,\alpha} x_i {\omega_{ij}}^\alpha \dot{y}_\alpha s_j \\
&= \sum_{j=1}^{k} \left(\sum_{\alpha=1}^{n} \left(\frac{\partial x_j}{\partial y_\alpha} \dot{y}_\alpha + \sum_{i=1}^{k} x_i {\omega_{ij}}^\alpha \right) \dot{y}_\alpha \right) s_j
\end{aligned}$$

Proposition 13.5. Let $E \to M$ be any smooth vector bundle, with a connection ∇. Let $c : [0,1] \to M$ be a smooth curve and $v \in E_{c(0)}$. Then there exists a unique lift $\tilde{c} : [0,1] \to E$ with $\pi \circ \tilde{c} = c$, with $\tilde{c}(0) = v$ and $\nabla_{\dot{c}} s \equiv 0$ if s is a section of $E\Big|_{\operatorname{im} c}$ given by $\tilde{c}$.

Proof. By compactness of $[0,1]$, we can choose a finite subdivision $t_0 = 0 < t_1 < \dots < t_l = 1$, s.t. $c\Big|_{[t_i, t_{i+1}]}$ has image in an open set in M, which is the domain of a chart and over which E is trivial.

Without loss of generality, we only need to prove the proposition for c with image in a chart where E is trivial.

We write $c(t) = (y_1(t), \dots, y_n(t))$ and use the frame $s_1, \dots, s_k$ given by the trivialization.

$\tilde{c}(t) = \sum_{i=1}^{k} x_i(t) s_i(c(t))$ with $v = \tilde{c}(0) = \sum_{i=1}^{k} x_i(0) s_i(c(0))$.

The equation $\nabla_{\dot{c}} s \equiv 0$ is equivalent to

$$\sum_{\alpha=1}^{n} \left(\frac{\partial x_j}{\partial y_\alpha} + \sum_{i=1}^{k} x_i {\omega_{ij}}^\alpha \right) \dot{y}_\alpha \equiv 0, \qquad \forall j \in \{1, \dots, k\}$$

$$\dot{x}_j + \sum_{i,\alpha} x_i {\omega_{ij}}^\alpha \equiv 0$$

This is a linear system of ODE for the function $x_1, \dots, x_k$.

For every initial condition $x_1(0), \dots, x_k(0)$, there is a unique smooth solution, which, moreover, depends linearly on the initial condition. ☐

Corollary 13.6. Let $E \xrightarrow{\pi} M$ and ∇ be as in the proposition. Then every smooth curve $c : [0,1] \to M$ defines a unique linear map

$$\begin{aligned}
E_{c(0)} &\to E_{c(1)} \\
\tilde{c}(0) = v &\mapsto \tilde{c}(1)
\end{aligned}$$

This linear map is an isomorphism

Let $E \xrightarrow{\pi} M$ be any smooth vector bundle over M. $p, q \in M$, E_p, E_q are k-dimensional vector spaces.

(1) If $p, q \in U \subset M$ and $\psi : E\Big|_U \to U \times \mathbb{R}^k$ trivialization, then ψ identifies E_p with $\{p\} \times \mathbb{R}^k$ and E_q with $\{q\} \times \mathbb{R}^k$, so those are identified using ψ.

(2) If a connection ∇ on E is given and there exists a smooth path $c(0) = p$, $c(1) = q$, then P_c =parallel transport along c defines an isomorphism between E_p and E_q. P_c depends not just on ∇ but also on c.

In a trivialization, every ∇ is given by $\nabla_\alpha = \partial_\alpha + A^\alpha$.

$$\omega_{ij} = \sum_\alpha {\omega_{ij}}^\alpha dy_\alpha, \qquad A^\alpha = {\omega_{ij}}^\alpha$$

Proposition 13.7. Let $E \xrightarrow{\pi} M$ be a smooth vector bundle of rank k and $s_1, \ldots, s_k$ be frame. Let ∇ be a connection on E, ω_{ij} its connection matrix with respect to $s_1, \ldots, s_k$ and Ω_{ij} its curvature matrix. If we pick a chart with coordinate functions $y_1, \ldots, y_n$, then

$$[\nabla_\alpha, \nabla_\beta] s_i = \sum_{j=1}^k \Omega_{ij}(\partial_\alpha, \partial_\beta) s_j$$

Corollary 13.8. $F^\nabla \equiv 0 \Leftrightarrow \Omega_{ij}$ for every local frame $\Leftrightarrow [\nabla_\alpha, \nabla_\beta] = 0$.

Proof.

$$\begin{aligned}
\Omega_{ij}(\partial_\alpha, \partial_\beta) &= \left(d\omega_{ij} - \sum_{l=1}^k \omega_{il} \wedge \omega_{lj} \right) (\partial_\alpha, \partial_\beta) \\
&= L_{\partial_\alpha}({\omega_{ij}}^\beta) - L_{\partial_\beta}({\omega_{ij}}^\alpha) - \sum_{l=1}^k {\omega_{il}}^\alpha {\omega_{lj}}^\beta - {\omega_{il}}^\beta {\omega_{lj}}^\alpha \\
&= \partial_\alpha {\omega_{ij}}^\beta - \partial_\beta {\omega_{ij}}^\alpha - \sum_{l=1}^k ({\omega_{il}}^\alpha {\omega_{lj}}^\beta - {\omega_{il}}^\beta {\omega_{lj}}^\alpha)
\end{aligned}$$

$$\begin{aligned}
\nabla_\alpha \nabla_\beta s_i &= \nabla_\alpha(\langle \nabla s_i, \partial_\beta \rangle) = \nabla_\alpha \left(\left\langle \sum_{j=1}^k \omega_{ij} \otimes s_j, \partial_\beta \right\rangle \right) \\
&= \nabla_\alpha \left(\sum_{j=1}^k {\omega_{ij}}^\beta s_j \right) = \left\langle \nabla \left(\sum_{j=1}^k \omega_{ij} s_j \right), \partial_\alpha \right\rangle \\
&= \sum_{j=1}^k \langle d{\omega_{ij}}^\beta \otimes s_j + {\omega_{ij}}^\beta \nabla s_j, \partial_\alpha \rangle \\
&= \sum_{j=1}^k (\partial_\alpha {\omega_{ij}}^\beta) s_j + \sum_{j=1}^k {\omega_{ij}}^\beta \sum_{l=1}^k {\omega_{jl}}^\alpha s_l \\
&= \sum_{j=1}^k \left(\partial_\alpha {\omega_{ij}}^\beta + \sum_{l=1}^k {\omega_{il}}^\beta \omega_{lj}^\alpha \right) s_j
\end{aligned}$$

$$\begin{aligned}\Rightarrow\qquad & (\nabla_\alpha\nabla_\beta-\nabla_\beta\nabla_\alpha)s_i\\ &=\sum_{j=1}^k\left(\partial_\alpha {\omega_{ij}}^\beta+\sum_{l=1}^k {\omega_{il}}^\beta {\omega_{lj}}^\alpha-\partial\ \ _\beta {\omega_{ij}}^\alpha-\sum_{l=1}^k {\omega_{il}}^\alpha {\omega_{lj}}^\beta\right)s_j\\ &=\sum_{j=1}^k\left(\partial_\alpha {\omega_{ij}}^\beta-\partial_\beta {\omega_{ij}}^\alpha-\sum_{l=1}^k\left({\omega_{il}}^\alpha {\omega_{lj}}^\beta-{\omega_{il}}^\beta {\omega_{lj}}^\alpha\right)\right)s_j\\ &=\sum_{j=1}^k\Omega_{ij}(\partial_\alpha,\partial_\beta)s_j\end{aligned}$$

□

Over a curve, every ∇ admits local trivializations by parallel sections, i.e. a parallel frame.

Theorem 13.9. *E admits a system of local trivializations by ∇-parallel frames if and only if $F^\nabla \equiv= 0$.*

Definition 13.4. $s\in\Gamma(E)$ is ∇-parallel if $\nabla s\equiv 0$. A frame $s_1,\dots,s_k$ is ∇-parallel if $\nabla s_i\equiv 0$, $\forall i$.

Proof. Suppose $s_1,\dots,s_k$ is a ∇-parallel local frame. Then $0=\nabla s_i=\sum_j\omega_{ij}\otimes s_j$ $\Rightarrow \omega_{ij}=0$, $\forall i,j$. $\Rightarrow \Omega_{ij}=d\omega_{ij}-\sum_l\omega_{il}\wedge\omega_{lj}=0$, so $F^\nabla\equiv 0$.

Conversely, if $F^\nabla\equiv 0$, we want to find local ∇-parallel frames. Since the statement is local, we work on $M=\mathbb{R}^n$.

For $n=1$, we find a ∇-parallel frame over $\mathbb{R}$ by parallel transport.

For $n>1$, we prove the statement by induction.

Let $p\geqslant 1$ and assume we have a ∇-parallel frame over $\mathbb{R}^p\times\{0\}\subset\mathbb{R}^{p+1}$. By construction, E is trivial on $\mathbb{R}^n$, so we may pick an arbitrary frame for E. We need to find a gauge transformation g, so that $s'_i=\sum_j g_{ij}s_j$ gives a ∇-parallel frame $s'_1,\dots,s'_k$. We want to solve

$$0=\omega'_{ij}=(dg\cdot g^{-1}+g\omega g^{-1})_{ij}$$

$$\begin{aligned}\Leftrightarrow\qquad && {\omega'_{ij}}^\alpha=0, &&\forall\alpha\\ \Leftrightarrow\qquad && {(dg+g\omega)_{ij}}^\alpha=0, &&\forall\alpha\\ \Leftrightarrow\qquad && \partial_\alpha g_{ij}+\sum_l g_{il}{\omega_{lj}}^\alpha=0, &&\forall\alpha \qquad (*)\end{aligned}$$

For the inductive step, we assume, we have a g s.t. (*) holds for $\alpha\leqslant p$.

In the inductive step, we assume the statement has been proved for $\mathbb{R}^p$. This means ${\omega_{ij}}^\alpha=0$ for $\alpha\leqslant p$.

To obtain the statement over $\mathbb{R}^{p+1}$, we need to solve

$$\partial_\alpha g_{ij}=0 \text{ for } \alpha\leqslant p \text{ and } \partial_{p+1}g_{ij}+\sum_l g_{il}{\omega_{lj}}^{p+1}=0\ \forall i,j \qquad (**)$$

Fix all y_β except y_{p+1}. We treat the second equation in (**) as an ODE in y_{p+1}. With initial condition $g(0) = \mathbb{1}$, this ODE has a unique solution.

Varying the starting point (the y-coordinates other than y_{p+1}), the solutions of the ODE vary smoothly.

The assumption that $F^\nabla \equiv 0$, means $[\nabla_\alpha, \nabla_\beta] = 0$, $\forall \alpha, \beta$. Take $\alpha \leqslant p$, $\beta = p+1$. Then

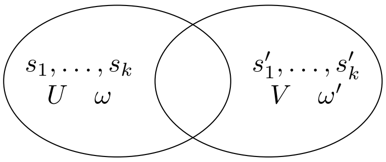
$$\partial_\alpha {\omega_{il}}^{p+1} - \partial_{p+1}\cancel{{\omega_{il}}^{\alpha}} + \sum_j ({\omega_{ij}}^{p+1}\cancel{{\omega_{jl}}^{\alpha}} - \cancel{{\omega_{ij}}^{\alpha}}{\omega_{jl}}^{p+1}) = 0$$

$$\begin{aligned} \Rightarrow \qquad & \partial_\alpha {\omega_{il}}^{p+1} = 0 \\ \Rightarrow \qquad & {\omega'_{ij}}^{\,p+1} = (dg\cdot g^{-1} + g\omega g^{-1})_{ij}^{p+1} = 0 \end{aligned}$$

because g solves the second equation in (**). ☐

Corollary 13.10. A vector bundle $E \xrightarrow{\pi} M$ admits a flat connection ∇ if and only if it admits a system of trivializations with constant transition maps.

Proof. If E admits ∇ with $F^\nabla \equiv 0$, then we can find local trivialization given by ∇-parallel frames.



$$\psi_U : \pi^{-1}(U) \to U \times \mathbb{R}^k \qquad \psi_V : \pi^{-1}(V) \to V \times \mathbb{R}^k$$

$$v = \sum_i \lambda_i s_i \mapsto (\pi(v), (\lambda_1, \dots, \lambda_k)) \qquad v = \sum_i \mu_i s'_i \mapsto (\pi(v), (\mu_1, \dots, \mu_k))$$

$$\begin{aligned} \psi_V \circ \psi_U^{-1} : (U \cap V) \times \mathbb{R}^k &\to (U \cap V) \times \mathbb{R}^k \\ (p, \omega) &\mapsto (p, g(p)\omega) \end{aligned}$$

where $g : U \cap V \to GL_k(\mathbb{R})$ is smooth.

$$\cancel{\omega'} = dgg^{-1} + g\cancel{\omega}g^{-1} \Leftrightarrow dg \equiv 0, \text{ so } g \text{ is constant.}$$

Conversely suppose we have (U_i, ψ_i) a system of local trivialization for E, s.t. each $\psi_j \circ \psi_i^{-1}$ has the form $(p, \omega) \mapsto (p, g(p)\omega)$ with g constant.

On $E\Big|_{U_i}$, we define a connection ∇ by making the constant sections in the trivial bundle parallel, i.e. $s_i(p) = \psi^{-1}(p, e_i)$.

$$\nabla\left(\sum_j f_j s_j\right) = \sum_j df_j \otimes s_j$$

☐

Claim 13.11. If $U_i \cap U_j \neq \varnothing$, then $\nabla^i = \nabla^j$ on $\pi^{-1}(U_i \cap U_j)$.

Proof. $s'_i = \sum_j g_{ij} s_j$, with $dg_{ij} \equiv 0$.
$\Rightarrow s'_i$ which are ∇^j parallel are also ∇^i parallel.
$\Rightarrow \nabla^i = \nabla^j$
$\Rightarrow$ The ∇^i fit together to a global connection ∇. Since ∇^i is flat, so is ∇. □

Remark. To prove existence of connections on arbitrary E, we also took local trivializations (U_i, ψ_i) and the corresponding flat connection ∇^i. If the transition functions are not constant, the ∇^i do not agree on the overlaps of their domains.

$\nabla = \sum_i \rho_i \nabla^i$, ρ_i a smooth partition of unity, is not flat.

13.6 Compatible

$E \xrightarrow{\pi} M$ admits a positive definite metric $\langle \ , \ \rangle : \Gamma(E) \times \Gamma(E) \to \mathcal{C}^\infty(M)$.

Definition 13.5. A connection ∇ on E is **compatible** with $\langle \ , \ \rangle$, if

$$d\langle s_1, s_2\rangle = \langle \nabla s_1, s_2\rangle + \langle s_1, \nabla s_2\rangle, \qquad \forall s_1, s_2 \in \Gamma(E) \tag{13.2}$$

Lemma 13.12. ∇ is compatible with $\langle \ , \ \rangle$ if and only if for every orthonormal local frame $s_1, \ldots, s_k$, the connection matrix ω representing ∇ is skew-symmetric, i.e. $\omega_{ij} = -\omega_{ji}$, $\forall i, j$.

Proof. Let $s_1, \ldots, s_k$ be orthonormal frame with respect to $\langle \ , \ \rangle$. Then

$$\langle s_i, s_j\rangle = \text{const.}, \qquad \forall i, j$$

If ∇ is compatible with $\langle \ , \ \rangle$, then

$$\begin{aligned} 0 &= d\langle s_i, s_j\rangle \\ &= \langle \nabla s_i, s_j\rangle + \langle s_i, \nabla s_j\rangle \\ &= \left\langle \sum_l \omega_{il} \otimes s_l, s_j \right\rangle + \left\langle s_j, \sum_l \omega_{jl} \otimes s_l \right\rangle \\ &= \sum_l (\omega_{il}\langle s_l, s_j\rangle + \omega_{jl}\langle s_i, s_l\rangle) \\ &= \omega_{ij} + \omega_{ji} \end{aligned}$$

$$\Leftrightarrow \qquad \omega_{ij} = -\omega_{ji}$$

Conversely, assume ω is skew-symmetric

$$\langle \nabla s_i, s_j\rangle + \langle s_i, \nabla s_j\rangle = \omega_{ij} + \omega_{ji} = 0$$

$$\langle s_i, s_j\rangle = \text{const.} \ \Rightarrow \ d\langle s_i, s_j\rangle = 0$$

$\Rightarrow$ (13.2) holds for the basis sections.

Let $s = \sum_i f_i s_i$ and $s' = \sum_j g_j s_j$. Then

$$\langle s, s'\rangle = \sum_i f_i g_i \ \Rightarrow \ d\langle s, s'\rangle = \sum_i f_i dg_i + \sum_i g_i df_i$$

$$\begin{aligned}\langle\nabla s, s'\rangle + \langle s, \nabla s'\rangle &= \sum_{i,j}\langle df_i\otimes s_i + f_i\nabla s_i, g_js_j\rangle + \langle f_is_i, dg_j\otimes s_j + g_j\nabla s_j\rangle\\ &= \sum_i g_idf_i + \sum_i f_idg_i + \sum_{i,j} f_ig_j\underbrace{(\langle\nabla s_i, s_j\rangle + \langle s_i,\nabla s_j\rangle)}_{=0 \text{ by above}}\end{aligned}$$

$$\Rightarrow \qquad d\langle s, s'\rangle = \langle\nabla s, s'\rangle + \langle s, \nabla s'\rangle$$

☐

Lemma 13.13. If ∇ is compatible with $\langle\ ,\ \rangle$, then Ω is skew-symmetric for every orthonormal frame.

Proof.

$$\begin{aligned}\Omega_{ij} &= d\omega_{ij} - \sum_l \omega_{il}\wedge\omega_{lj}\\ &= -d\omega_{ji} - \sum_l \omega_{li}\wedge\omega_{jl}\\ &= -(d\omega_{ij} - \sum_l \omega_{jl}\wedge\omega_{li})\\ &= -\Omega_{ji}\end{aligned}$$

☐

Definition 13.6. $A\in\Gamma(\operatorname{End}E)$ is skew-symmetric with respect to $\langle\ ,\ \rangle$ if

$$\langle As, s'\rangle = -\langle s, As'\rangle, \qquad \forall s, s'\in\Gamma(E)$$

$\operatorname{End}E = \mathrm{Skew}-\operatorname{End}E\oplus\mathrm{Sym}-\operatorname{End}(E)$.

Proposition 13.14. For every metric $\langle\ ,\ \rangle$, there exist compatible connections ∇. All such connections is naturally an affine space for $\Omega^1(\mathrm{Skew}-\operatorname{End}(E))$.

Proof. Let $\{U_i \mid i\in I\}$ be an open cover of M, s.t. $E\Big|_{U_i}$ is trivial. Then on every U_i, we have an orthonormal local frame for E with respect to $\langle\ ,\ \rangle$. Let $s_1,\dots,s_k$ be such an orthonormal frame over U_i. Define ∇^i on $E\Big|_{U_i}$ by $\nabla^i(s_j)\equiv 0$.

Claim 13.15. ∇^i is compatible with $\langle\ ,\ \rangle$.

Proof. With respect to orthonormal frame $s_1,\dots,s_k$, $\omega_{ij}\equiv 0$. So ω_{ij} is skew-symmetric. Let ρ_i be a smooth partition of unity subordinate to U_i. ☐

Define $\nabla := \sum\limits_i \rho_i\nabla^i$. This is a connection on E compatible with $\langle\ ,\ \rangle$, because each ∇^i is.

Suppose ∇, ∇' are both compatible with $\langle\ ,\ \rangle$. Set $\nabla-\nabla' = A\in\Omega^1(\operatorname{End}E)$. Then

$$\begin{aligned}\langle As, s'\rangle &= \langle(\nabla-\nabla')s, s'\rangle\\ &= \langle\nabla s, s'\rangle - \langle\nabla' s, s\rangle\\ &= d(\langle s, s'\rangle) - \langle s, \nabla s'\rangle - d(\langle s, s'\rangle) + \langle s, \nabla' s\rangle\\ &= -\langle s, (\nabla-\nabla')s'\rangle\\ &= -\langle s, As'\rangle\end{aligned}$$

So $A \in \Omega^1(\mathrm{Skew}-\mathrm{End}(E))$.

If ∇ is compatible with the metric and $A \in \Omega^1(\mathrm{Skew}-\mathrm{End}\,E)$, then $\nabla + A$ is also compatible. ☐

Example 13.1. $k = 1$: Let s be a (local) section of E, s nowhere zero.

$$\nabla s = \alpha \otimes s = \omega_{11} \otimes s$$

$$\Omega_{11} = d\omega_{11} - \sum_l \omega_{1l} \wedge \omega_{l1} = d\omega_{11} - \cancel{\omega_{11} \wedge \omega_{11}}$$

$\Rightarrow$
$$d\Omega_{11} = 0$$

Suppose we have a metric $\langle\ ,\ \rangle$ and $\langle s, s\rangle \equiv 1$. If ∇ is compatible with $\langle\ ,\ \rangle$, then $\nabla s \equiv 0$.

$$\langle s, s\rangle = \text{const.} \;\Rightarrow\; 0 = 2\langle s, \nabla s\rangle \;\Rightarrow\; \nabla s \equiv 0, \text{by 1-dimension}$$

Conclusion: Every compatible connection ∇ on a rank 1 bundle is flat. $\Rightarrow$ Every rank 1 bundle admits a flat connection.

13.7 Affine Connection

Definition 13.7. A connection on $E = TM$ is called an **affine connection** on M.

$$\begin{array}{ccccc} \Gamma(E) & \to & \Omega^1(E) & \xrightarrow{i_X} & \Gamma(E) \\ s & \mapsto & \nabla s & \mapsto & \nabla_X s \end{array}$$

where $X \in \mathfrak{X}(M)$. If $E = TM$, then $s \in \mathfrak{X}(M)$.

Example 13.2. There is no affine connection ∇ satisfying $\nabla_X Y = \nabla_Y X$, $\forall X, Y \in \mathfrak{X}(M)$.

13.8 Torsion

Definition 13.8. If ∇ is an affine connection on M, then

$$T^\nabla(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y], \qquad \text{for } X, Y \in \mathfrak{X}(M)$$

T^∇ is the **torsion** of ∇.

Definition 13.9. ∇ is **symmetric** if it is torsion-free, i.e. $T^\nabla \equiv 0$.

Proposition 13.16 (Properties of T^∇)**.**

(1) T^∇ is skew-symmetric in X, Y.

(2) T^∇ is $\mathcal{C}^\infty(M)$-linear in X and Y.

Proof.

$$\begin{aligned} T^\nabla(fX, Y) &= \nabla_{fX} Y - \nabla_Y fX - [fX, Y] \\ &= f\nabla_X Y - \langle df \otimes X + f\nabla X, Y\rangle - f[X,Y] + \underbrace{L_Y f}_{=df(Y)} X \\ &= f \cdot T^\nabla(X,Y) \end{aligned}$$

☐

Let $x_1, \dots, x_n$ be local coordinates on M given by some charts (U, φ). Then $\partial_1, \dots, \partial_n$ form a local frame for $TM\Big|_U = TU$. If ∇ is any affine connection on M, we can write

$$\nabla \partial_i = \sum_{j=1}^n \omega_{ij} \otimes \partial_j \qquad \omega_{ij} = \sum_{l=1}^n {\omega_{ij}}^l dx_l$$

$$\nabla_{\partial_l} \partial_i = \sum_{j=1}^n \langle \omega_{ij}, \partial_l \rangle \partial_j = \sum_{j=1}^n {\omega_{ij}}^l \partial_j = \sum_{j=1}^n \Gamma_{li}^j \partial_j$$

The Γ_{li}^j are called the Christoffel symbols of ∇ with respect to the local coordinates $x_1, \dots, x_n$

$$T^\nabla(\partial_\alpha, \partial_\beta) = \nabla_{\partial_\alpha} \partial_\beta - \nabla_{\partial_\beta} \partial_\alpha - \cancel{[\partial_\alpha, \partial_\beta]} = \sum_{j=1}^n (\Gamma_{\alpha\beta}^j - \Gamma_{\beta\alpha}^j) \partial_j$$

Lemma 13.17. $T^\nabla \equiv 0 \Leftrightarrow \Gamma_{\alpha\beta}^j = \Gamma_{\beta\alpha}^j$, $\forall \alpha, \beta, j \in \{1, \dots, n\}$ and all local coordinate systems on M.

Definition 13.10. ∇^* on E^* by

$$\begin{aligned} d\lambda(s) &= \lambda(\nabla s) + (\nabla^* \lambda)(s) \\ d\langle \lambda, s \rangle &= \langle \lambda, \nabla s \rangle + \langle \nabla^* \lambda, s \rangle \end{aligned}$$

where $\forall \lambda \in \Gamma(E^*), s \in \Gamma(E)$.

Claim 13.18. ∇^* is a connection on E^*.

Proof.

$$(\nabla^* \lambda)(s) = d\lambda(s) - \lambda(\nabla(s))$$

$$\begin{aligned} (\nabla^*(f\lambda))(s) &= d(f\lambda)(s) - (f\lambda)(\nabla s) \\ &= d(f \cdot \lambda(s)) - (f \cdot \lambda)(\nabla s) \\ &= \lambda(s) df + f d\lambda(s) - f \cdot \lambda(\nabla(s)) \\ &= \lambda(s) df + f(\nabla^* \lambda)(s) \\ &= (df \cdot \lambda + f \nabla^* \lambda)(s) \end{aligned}$$

Let $s_1, \dots, s_k$ be a local frame for E, and $\lambda_1, \dots, \lambda_k$ the dual frame for E^*, i.e.

$$\begin{aligned} \lambda_i(s_j) &= \delta_{ij} \\ 0 &= \lambda_i(\nabla s_j) + (\nabla^* \lambda_i)(s_j) \\ &= \lambda_i \left(\sum_{m=1}^k \omega_{jm} \otimes s_m \right) + \left(\sum_{m=1}^k \omega_{il}^* \otimes \lambda_l \right) (s_j) \\ &= \omega_{ji} + \omega_{ij}^* \end{aligned}$$

$$\Rightarrow \qquad \omega_{ij}^* = -\omega_{ji}, \quad \omega^* = -\omega^t$$

If ∇ is an affine connection of M, then ∇^* is a connection on T^*M. ☐

Proposition 13.19. ∇ is torsion-free if and only if

$$\begin{tikzcd}
\Omega^1(M) = \Gamma(T^*M) \arrow[r, "\nabla^*"] \arrow[rr, bend right, "=d"'] & \Omega^1(T^*M) = \Gamma(T^*M \otimes T^*M) \arrow[r, "\sim"] & \Omega^2(M)
\end{tikzcd}$$

Proof. Let $x_1, \dots, x_n$ be local coordinates, given by a chart (U, φ). Then $\partial_1, \dots, \partial_n$ is a local frame for TM and $dx_1, \dots, dx_n$ is the dual frame for T^*M.

Every 1-form α on U is of the form

$$\beta = \sum_{i=1}^{n} f_i dx_i$$

$$\Rightarrow \quad d\beta = \sum_{i=1}^{n} df_i \wedge dx_i = \sum_i \sum_j \frac{\partial f_i}{\partial x_j} dx_j \wedge dx_i = \sum_{i<j} \left(\frac{\partial f_i}{\partial x_j} - \frac{f_j}{\partial x_i} \right) dx_j \wedge dx_i$$

$$\begin{aligned}
\nabla^*\beta &= \sum_i \nabla^*(f_i dx_i) \\
&= \sum_i df_i \otimes dx_i + f_i \nabla^* dx_i \\
&= \sum_i df_i \otimes dx_i + f_i \sum_j \omega^*_{ij} \otimes dx_j \\
&= \sum_i \left(df_i \otimes dx_i - f_i \sum_j \omega_{ji} \otimes dx_j \right) \\
&= \sum_i \left(\sum_j \frac{\partial f_i}{\partial x_j} dx_j \otimes dx_i - f_i \sum_{j,\alpha} {\omega_{ji}}^{\alpha} dx_\alpha \otimes dx_j \right) \\
&= \sum_{i,j} \frac{\partial f_i}{\partial x_j} dx_j \otimes dx_i - \sum_{i,j,\alpha} f_i {\omega_{ji}}^{\alpha} dx_\alpha \otimes dx_j \\
&= \sum_{j,\alpha} \frac{\partial f_j}{\partial x_\alpha} dx_\alpha \otimes dx_j - \sum_{i,j,\alpha} f_i {\omega_{ji}}^{\alpha} dx_\alpha \otimes dx_j \\
&= \sum_{j,\alpha} \left(\frac{\partial f_j}{\partial x_\alpha} - \sum_i f_i {\omega_{ji}}^{\alpha} \right) dx_\alpha \otimes dx_j \\
\wedge(\nabla^*\beta) &= \sum_{j<\alpha} \left(\frac{\partial f_j}{\partial x_\alpha} - \frac{\partial f_\alpha}{\partial x_j} - \sum_i f_i ({\omega_{ji}}^{\alpha} - {\omega_{\alpha i}}^{j}) \right) dx_\alpha \wedge dx_j
\end{aligned}$$

$$\wedge(\nabla^*\beta) = d\beta, \forall \beta \;\Leftrightarrow \omega^{\alpha}_{ji} - \omega^{j}_{\alpha i} = 0, \forall j, \alpha \;\Leftrightarrow \mathit{\Gamma}^i_{\alpha j} = \mathit{\Gamma}^i_{j\alpha}, \forall \alpha, j \;\Leftrightarrow T^{\nabla} \equiv 0$$

$\beta = df$:

$$\begin{aligned}
\nabla^*\beta &= \nabla^*\left(\sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i\right)\\
&= \sum_{i=1}^n d\left(\frac{\partial f}{\partial x_i}\right) dx_i + \frac{\partial f}{\partial x_i}\nabla^*(dx_i)\\
&= \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_j \otimes dx_i - \frac{\partial f}{\partial x_i}\omega_{ji} \otimes dx_j\\
&= \sum_{i,j}\left(\frac{\partial^2 f}{\partial x_i \partial x_j} dx_j \otimes dx_i - \sum_\alpha \frac{\partial f}{\partial x_i}{\omega_{ji}}^\alpha dx_\alpha \otimes dx_j\right)\\
&= \sum_{\alpha,j}\left(\frac{\partial^2 f}{\partial x_j \partial x_\alpha} - \sum_i \frac{\partial f}{\partial x_i}\Gamma^i_{\alpha j}\right) dx_\alpha \otimes dx_j
\end{aligned}$$

$T^\nabla \equiv 0 \Leftrightarrow \underbrace{\nabla^* df}_{\in\Gamma(T^*M\otimes T^*M)}$ is symmetric $\forall f \in \mathcal{C}^\infty(M)$.

$$\begin{aligned}
T^{\nabla+A}(X,Y) &= (\nabla + A)_X Y - (\nabla + A)_Y X - [X,Y]\\
&= \underbrace{\nabla_X Y - \nabla_Y X - [X,Y]}_{=T^\nabla} + A_X(Y) - A_Y(X)
\end{aligned}$$

where $A \in \Omega_1(\mathrm{End}(TM))$, $A_X \in \Gamma(\mathrm{End}(TM))$, $A_X Y$ is evaluation of the endomorphism A_X on Y. ☐

13.9 Riemannian Geometry

Theorem 13.20. Let $\langle\ ,\ \rangle$ be a metric on TM (a Riemannian metric on M). For every $\mathcal{C}^\infty(M)$-bilinear skew-symmetric

$$T : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

There exists a unique affine connection ∇ compatible with $\langle\ ,\ \rangle$ and $T^\nabla = T$.

Proof. Uniqueness: Suppose ∇ is compatible, with $\langle\ ,\ \rangle$ and $T^\nabla = T$.

$$\begin{aligned}
d\langle X,Y\rangle &= \langle \nabla X, Y\rangle + \langle X, \nabla Y\rangle, \qquad \forall X,Y \in TM\\
L_Z\langle X,Y\rangle &= \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle, \qquad \forall X,Y,Z \in TM\\
T(Z,Y) &= \nabla_Z Y - \nabla_Y Z - [Z,Y]
\end{aligned}$$

$$\begin{aligned}
\langle \nabla_Z X, Y\rangle &= L_Z\langle X,Y\rangle - \langle X, \nabla_Z Y\rangle\\
&= L_Z\langle X,Y\rangle - \langle X, T(Z,Y)\rangle - \langle X, \nabla_Y Z\rangle - \langle X, [Z,Y]\rangle\\
&= L_Z\langle X,Y\rangle - \langle X, T(Z,Y)\rangle - L_Y\langle X,Z\rangle + \langle \nabla_Y X, Z\rangle - \langle X, [Z,Y]\rangle\\
&= L_Z\langle X,Y\rangle - \langle X, T(Z,Y)\rangle - L_Y\langle X,Z\rangle + \langle T(Y,X), Z\rangle + \langle \nabla_X Y, Z\rangle\\
&+ \langle [Y,X], Z\rangle - \langle X, [Z,Y]\rangle\\
&= L_Z\langle X,Y\rangle - \langle X, T(Z,Y)\rangle - L_Y\langle X,Z\rangle + \langle T(Y,X), Z\rangle + L_X\langle Y,Z\rangle\\
&- \langle Y, \nabla_X Z\rangle + \langle [Y,X], Z\rangle - \langle X, [Z,Y]\rangle\\
&= L_Z\langle X,Y\rangle - \langle X, T(Z,Y)\rangle - L_Y\langle X,Z\rangle + \langle T(Y,X), Z\rangle + L_X\langle Y,Z\rangle\\
&- \langle Y, T(X,Z)\rangle - \langle Y, \nabla_Z X\rangle - \langle Y, [X,Z]\rangle + \langle [Y,X], Z\rangle - \langle X, [Z,Y]\rangle
\end{aligned}$$

Therefore, we have the so called **Koszul formula**.

$$\boxed{\begin{aligned}\langle \nabla_Z X, Y\rangle = \frac{1}{2}(&L_Z\langle X,Y\rangle - L_Y\langle X,Z\rangle + L_X\langle Y,Z\rangle - \langle X, T(Z,Y)\rangle + \langle Z, T(Y,X)\rangle \\ &- \langle Y, T(X,Z)\rangle - \langle Y,[X,Z]\rangle + \langle [Y,X],Z\rangle - \langle X,[Z,Y]\rangle)\end{aligned}}$$

This shows $\nabla_Z X$ is uniquely determined $\forall Z, X \in \mathfrak{X}(M)$.

Existence: Define $\nabla_Z X$ by the Koszul formula. Fix M and $\langle\ ,\ \rangle$ on TM. Let ∇ be the Levi-Civita connection of $\langle\ ,\ \rangle$. Let $x_1, \ldots, x_n$ be local coordinates given by a chart $\partial_1, \ldots, \partial_n$ the coordinate vector fields.

$$\begin{aligned}\gamma_{ij} &= \langle \partial_i, \partial_j\rangle \\ \langle \nabla_{\partial_i}\partial_j, \partial_k\rangle &= \frac{1}{2}(L_{\partial_i}\gamma_{jk} + L_{\partial_j}\gamma_{ki} - L_{\partial_k}\gamma_{ij}) = \frac{1}{2}(\partial_i\gamma_{jk} + \partial_j\gamma_{ki} - \partial_k\gamma_{ij}) \\ \nabla_{\partial_i}\partial_j &= \sum_{k=1}^n {\omega_{jk}}^i \partial_k = \sum_{k=1}^n \mathit{\Gamma}_{ij}^k \partial_k \\ \langle \nabla_{\partial_i}\partial_j, \partial_l\rangle &= \sum_{k=1}^n \mathit{\Gamma}_{ij}^k \gamma_{kl} = \frac{1}{2}(\partial_i\gamma_{jk} + \partial_j\gamma_{ki} - \partial_k\gamma_{ij})\end{aligned}$$

Formula of $\mathit{\Gamma}_{ij}^k$ in terms of γ_{ij}. □

Setting $T = 0$, we get

Corollary 13.21 (Fundamental Lemma of Riemannian Geometry)**.** For every metric on TM, there exists a unique, compatible, torsion-free connection.

Definition 13.11. This connection ∇ as in the corollary is called the **Levi-Civita connection** of $(M; \langle\ ,\ \rangle)$.

Definition 13.12. If ∇ is the Levi-Civita connection, then

$$R(X,Y)Z := (F^\nabla(X,Y))Z$$

is called the Riemann curvature tensor of the metric $\langle\ ,\ \rangle$.

This is trilinear over $\mathcal{C}^\infty(M)$.

$$\begin{aligned}R: \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) &\to \mathfrak{X}(M) \\ (X,Y,Z) &\mapsto R(X,Y)Z\end{aligned}$$

Equivalently, we can consider R as follows:

$$\begin{aligned}R: \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) &\to \mathcal{C}^\infty(M) \\ (X,Y,Z,W) &\mapsto \langle R(X,Y)Z, W\rangle\end{aligned}$$

Proposition 13.22 (Symmetries of R)**.**

(1) $R(X,Y)Z = -R(Y,X)Z$, because F^∇ is a 2-form.

(2) $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$, $\forall X, Y, Z$. Sometimes it is called the **first Bianchi Identity**.

(3) $\langle R(X,Y)Z,W\rangle = -\langle R(X,Y)W,Z\rangle$, $\forall X,Y,Z,W$.

(4) $\langle R(X,Y)Z,W\rangle = \langle R(Z,W)X,Y\rangle$, $\forall X,Y,Z,W$

Proof. (2) It is enough to prove (2) for X,Y,Z with pairwise vanishing brackets.

$$F^{\nabla}(X,Y)s = \nabla_X\nabla_Y s - \nabla_Y\nabla_X s - \cancel{\nabla_{[X,Y]}s}$$

In this case, left hand side of (2)

$$\begin{aligned}
&\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z + \nabla_Y\nabla_Z X - \nabla_Z\nabla_Y X + \nabla_Z\nabla_X Y - \nabla_X\nabla_Z Y \\
&= \nabla_X \underbrace{(\nabla_Y Z - \nabla_Z Y)}_{=0} + \nabla_Y \underbrace{(\nabla_Z X - \nabla_X Z)}_{=0} + \nabla_Z \underbrace{(\nabla_X Y - \nabla_Y X)}_{=0 \text{ since } T^{\nabla}=0} \\
&= 0
\end{aligned}$$

(3): We need to prove $\langle R(X,Y)Z,Z\rangle = 0$, $\forall X,Y,Z$. We may assume that X,Y,Z have vanishing brackets.

$$\langle R(X,Y)Z,Z\rangle = \langle \nabla_X\nabla_Y Z, Z\rangle - \langle \nabla_Y\nabla_X Z, Z\rangle$$

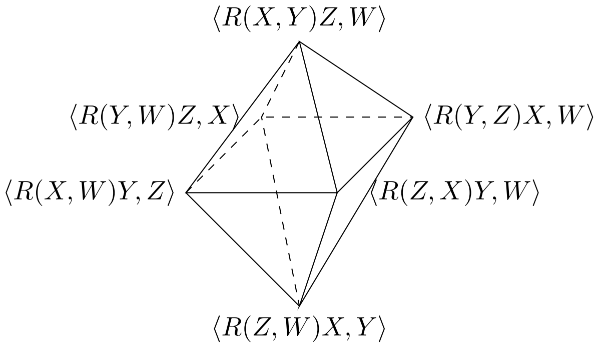
Consider

$$\begin{aligned}
L_X\langle Z,Z\rangle &= \langle \nabla_X Z, Z\rangle + \langle Z, \nabla_X Z\rangle = Z\langle \nabla_X Z, Z\rangle \\
L_Y L_X\langle Z,Z\rangle &= 2L_Y\langle \nabla_X Z, Z\rangle = 2(\langle \nabla_Y\nabla_X Z, Z\rangle + \langle \nabla_X Z, \nabla_Y Z\rangle)
\end{aligned}$$

$L_Y L_X\langle Z,Z\rangle$ is symmetric in X,Y, since $\langle X,Y\rangle = 0$ and $\langle \nabla_X Z, \nabla_Y Z\rangle$ is symmetric in X,Y. Therefore, $\langle \nabla_Y\nabla_X Z, Z\rangle$ is symmetric in X,Y. Thus

$$\Rightarrow \qquad \langle R(X,Y)Z,Z\rangle = 0$$

(4):



Sum for upper left-hand face is $\langle R(Y,X)W,Z\rangle + \langle R(W,Y)X,Z\rangle + \langle R(X,W)Y,Z\rangle$. Sum of labels is $= 0$ by (1)+(2)+(3) for top left and right and bottom front and back faces.

Sum the top left and right and subtract the bottom front and back faces: $\Rightarrow$

$$\begin{aligned}
&\Rightarrow \qquad \text{The middle nodes cancel} \\
&\Rightarrow \qquad 0 = \langle R(X,Y)Z,W\rangle - \langle R(Z,W)X,Y\rangle
\end{aligned}$$

□

Let M with metric and R its Riemann tensor.

Definition 13.13. Take $p \in M,\ X, Y \in T_pM$ linearly independent

$$K(X,Y) := \frac{\langle R(X,Y)Y, X\rangle}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2}$$

This is called the **sectional curvature** of $(M, \langle\ ,\ \rangle)$ with respect to the plane σ spanned by X, Y in T_pM.

Claim 13.23. $K(X,Y)$ depends only on $\sigma = \operatorname{span}\{X,Y\}$.

Proof. $K(\lambda X, Y) = \dfrac{\lambda^2\langle R(X,Y)Y,X\rangle}{\lambda^2 \cdot (\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2)} = K(X,Y) \neq 0.$

Since $K(X,Y) = K(Y,X)$, we also get $K(X, \lambda Y) = K(X,Y)$.

$$\begin{aligned} K(X, Y+\lambda X) &= \frac{\langle R(X,Y)(Y+\lambda X), X\rangle + \langle R(X,\lambda X)(Y+\lambda X), X\rangle}{\langle X,X\rangle(\langle Y,Y\rangle + \lambda^2\langle X,X\rangle + 2\lambda\langle X,Y\rangle) - \langle X, Y+\lambda X\rangle^2} \\ &= K(X,Y) \end{aligned}$$

This shows $K(X,Y)$ is the same $\forall X, Y \in \sigma$. ☐

Proposition 13.24. The collection of all sectional curvatures determines R.

Proof. Let V be a vector space with positive definite $\langle\ ,\ \rangle$.

Let $R, R' : V \times V \times V \to V$ be two trilinear maps satisfying the symmetry of the curvature tensor. Then if $K(X,Y) = \dfrac{\langle R(X,Y)Y,X\rangle}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2}$ equals K' computed in the same way from R' for all linear independent X, Y, $R = R'$.

$R(X,Y)Z = 0 = R'(X,Y)Z$, if X, Y are linear independent.

Assume X, Y linearly independent, then $K(X,Y) = K'(X,Y)$ implies

$$\langle R(X,Y)Y, X\rangle = \langle R'(X,Y)Y, X\rangle, \qquad \forall X, Y \text{ linearly independent}$$

$$\begin{aligned} \Rightarrow\quad & \langle R(X+Z,Y)Y, X+Z\rangle = \langle R'(X+Z,Y)Y, X+Z\rangle \\ \Leftrightarrow\quad & \cancel{\langle R(X,Y)Y,X\rangle} + \langle R(X,Y)Y,Z\rangle + \underbrace{\langle R(Z,Y)Y,Z\rangle}_{=\langle R(Y,Z)X,Y\rangle = \langle R(X,Y)Y,Z\rangle} + \cancel{\langle R(Z,Y)Y,Z\rangle} = (R \leftrightarrow R') \\ \Leftrightarrow\quad & 2\langle R(X,Y)Y,Z\rangle = 2\langle R'(X,Y)Y,Z\rangle, \qquad \forall Z \end{aligned}$$

After one more polarization $Y \mapsto Y + W$, we conclude

$$\langle R(X,Y)Z, W\rangle = \langle R'(X,Y)Z, W\rangle, \qquad \forall X, Y, Z, W$$

$$\Rightarrow \qquad R = R'$$

☐

Example 13.3. Let $M = \mathbb{R}^n$, and $\langle\ ,\ \rangle$ constant, standard. $\nabla\dfrac{\partial}{\partial x_n} = 0$ gives Levi-Civita $\Rightarrow R \equiv 0$, so $K \equiv 0$.

Example 13.4. Let $M \subset \mathbb{R}^{n+1}$ be smooth hypersurface. $\langle\ ,\ \rangle$ on $\mathbb{R}^{n+1}$ as in 13.3. ∇ the Levi-Civita connection of $\mathbb{R}^{n+1}$. We restrict the constant scalars product on $\mathbb{R}^{n+1}$ to the tangent space of M to get a metric $\langle\ ,\ \rangle$ on TM.

$$\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}\Big|_M = T\mathbb{R}^{n+1}\Big|_M = TM \oplus TM^\perp$$

where $TM^\perp$ is the normal bundle of M.

If M is orientable, then there is a uniquely defined unit normal vector field to M, so that the orientation of M together with the positive or of $\mathbb{R}$ defines the standard orientation of $\mathbb{R}^{n+1}$.

Definition 13.14. $G : M \to S^n \subset \mathbb{R}^{n+1}$ is the **Gauss map** of M.
$p \mapsto n(p)$

Definition 13.15. $L : T_pM \to T_pM$ is the **Weingarten map** of M at p.
$v \mapsto (\tilde{\nabla}_v n)(p)$

Lemma 13.25. L is self adjoint with respect to $\langle\ ,\ \rangle$.

Proof. Let $X, Y \in \mathfrak{X}(M)$.

$$\begin{aligned}
\langle L(X), Y\rangle &= \langle \tilde{\nabla}_X n, Y\rangle \\
&= L_X \cancel{\langle n, Y\rangle} - \langle n, \tilde{\nabla}_X Y\rangle \\
&= -\langle n, \tilde{\nabla}_Y X + \cancel{[X, Y]}\rangle \\
&= -\langle n, \tilde{\nabla}_Y X\rangle \\
&= -L_Y \cancel{\langle n, X\rangle} + \langle \tilde{\nabla}_Y n, X\rangle \\
&= \langle L(Y), X\rangle \\
&= \langle X, L(Y)\rangle
\end{aligned}$$

□

Lemma 13.26. $D_pG = L$.

Proof. $D_pG : T_pM \to T_{G(p)}S^n = T_pM$, since both are orthogonal complement of n.

Let $c : (-\varepsilon, \varepsilon) \to M$ be a smooth curve, with $c(0) = p$ and $\dot{c}(0) = v$. Then

$$\begin{aligned}
D_pG(v) &= (D_{c(0)}G)(\dot{c}(0)) \\
&= D_0(G \circ c)\left(\frac{\partial}{\partial t}\right) \\
&= \frac{d}{dt} n(c(t))\Big|_{t=0} \\
&= \tilde{\nabla}_{\dot{c}(0)} n \\
&= L(v)
\end{aligned}$$

□

Let X, $Y \in \mathfrak{X}(M)$.

$$\tilde{\nabla}_X Y = (\tilde{\nabla}_X Y)_t + (\tilde{\nabla}_X Y)_n \quad \text{with respect to } \mathbb{R}^{n+1} = T_pM \oplus \mathbb{R}n(p)$$

Definition 13.16. Define $\nabla_X Y = \pi(\tilde{\nabla}_X Y)$, $\pi : \mathbb{R}^{n+1} \to T_pM$ is the projection with kernel $\mathbb{R}n(p)$.

Lemma 13.27. ∇ is the Levi Civita connection of M.

Proof. Step 1: ∇ is a connection on TM. $\nabla_X Y$ is $\mathbb{R}$-linear in X, Y and it is $\mathcal{C}^\infty(M)$-linear in X.

$$\nabla_X(fY) = \pi(\tilde{\nabla}_X(fY)) = \pi(L_X f \cdot Y + f\tilde{\nabla}_X Y) = L_X f \cdot Y + f \cdot \nabla_X Y$$

Leibniz rule for ∇.

Step 2: ∇ on TM is compatible with $\langle\ ,\ \rangle$.

$$\langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle = \langle \tilde{\nabla}_X Y, Z\rangle + \langle Y, \tilde{\nabla}_X Z\rangle = L_X\langle Y, Z\rangle, \qquad X, Y, Z \in \mathfrak{X}(M)$$

Step 3:

$$0 = T^{\tilde{\nabla}}(X, Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y], \qquad X, Y \in \mathfrak{X}(M) \tag{13.3}$$

projecting to TM gives

$$0 = \nabla_X Y - \nabla_Y X - [X, Y] = T^{\nabla}(X, Y)$$

In (13.3), take $\langle -, n\rangle$

$$0 = \langle \tilde{\nabla}_X Y, n\rangle - \langle \tilde{\nabla}_Y X, n\rangle \xlongequal{\text{Lemma 1 Proof}} -\langle L(X), Y\rangle + \langle X, L(Y)\rangle$$

$$\Leftrightarrow \qquad L \text{ is self adjoint with respect to } \langle\ ,\ \rangle.$$

□

$X, Y \in \mathfrak{X}(M)$, $\tilde{\nabla}_X Y = \nabla_X Y + \langle \tilde{\nabla}_X Y, n\rangle n = \nabla_X Y - \langle L(X), Y\rangle n$.
Take $X, Y, Z \in \mathfrak{X}(M)$.

$$\begin{aligned}\tilde{\nabla}_X\tilde{\nabla}_Y Z &= \tilde{\nabla}_X(\nabla_Y Z - \langle L(Y), Z\rangle n) \\ &= \tilde{\nabla}_X\nabla_Y Z - L_X\langle L(Y), Z\rangle \cdot n - \langle L(Y), Z\rangle\tilde{\nabla}_X n \\ &= \nabla_X\nabla_Y Z - \langle L(X), \nabla_Y Z\rangle \cdot n - \langle \nabla_X L(Y), Z\rangle \cdot n - \langle L(Y), \nabla_X Z\rangle \cdot n \\ &- \langle L(\nabla_X Y), Z\rangle \cdot n - \langle L(Y), Z\rangle L(X)\end{aligned}$$

Similarly for $\tilde{\nabla}_Y\tilde{\nabla}_X Z$

$$\tilde{\nabla}_{[X,Y]}Z = \nabla_{[X,Y]}Z - \langle L([X, Y], Z)\rangle n$$

$$\begin{aligned}0 = \tilde{R}(X, Y)Z &= \tilde{\nabla}_X\tilde{\nabla}_Y Z - \tilde{\nabla}_Y\tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]}Z \\ &= \nabla_X\nabla_Y Z - \langle L(Y), Z\rangle L(X) - (\langle L(X), \nabla_Y Z\rangle + \langle \nabla_X L(Y), Z\rangle + \langle L(Y), \nabla_X Z\rangle)n \\ &- \nabla_Y\nabla_X Z + \langle L(X), Z\rangle L(Y) + (\langle L(Y), \nabla_X Z\rangle + \langle \nabla_Y L(X), Z\rangle + \langle L(X), \nabla_Y Z\rangle)n \\ &- \nabla_{[X,Y]}Z + \langle L([X, Y]), Z\rangle n\end{aligned}$$

Projecting to TM, we get the **Gauss equation**

$$\Rightarrow \qquad R(X, Y)Z = \langle L(Y), Z\rangle L(X) - \langle L(X), Z\rangle L(Y)$$

Projecting to n

$$\begin{aligned}
\Rightarrow \quad 0 &= \cancel{-\langle L(X), \nabla_Y Z\rangle} - \langle \nabla_X L(Y), t\rangle \cancel{-\langle L(Y), \nabla_X Z\rangle} + \cancel{\langle L(Y), \nabla_X Z\rangle} \\
&\qquad + \langle \nabla_Y L(X), Z\rangle + \cancel{\langle L(X), \nabla_Y Z\rangle} + \langle L([X,Y]), Z\rangle \\
\Rightarrow \quad \langle L([X,Y]), Z\rangle &= \langle \nabla_X L(Y), Z\rangle - \langle \nabla_Y L(X), Z\rangle \qquad \forall X, Y, Z \in \mathfrak{X}(M) \\
\Rightarrow \quad L([X,Y]) &= \nabla_X L(Y) - \nabla_Y L(X) \qquad \forall X, Y \in \mathfrak{X}(M)
\end{aligned}$$

This is called the Codazzi-Mainardi equation. We can apply the Gauss equation to any smooth hypersurface $M \subset \mathbb{R}^{n+1}$. If M is an affine hyperplane, G is constant, so $L \equiv DG \equiv 0 \Rightarrow R(X,Y)Z = 0$.

If $M \subset \mathbb{R}^{n+1}$ is the unit sphere S^n, then $G = \mathrm{Id} \Rightarrow L = DG = \mathrm{Id}$. By the Gauss equation

$$R(X,Y)Z = \langle Y, Z\rangle X - \langle X, Z\rangle Y$$

$X, Y \in T_p S^n$, linear independent:

$$K(X,Y) = \frac{\langle R(X,Y)Y, X\rangle}{\langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2} = \frac{\langle Y, Y\rangle\langle X, X\rangle - \langle X, Y\rangle\langle Y, X\rangle}{\langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2} = 1$$

If $M = S^n(r)$ is the sphere of Radius r in $\mathbb{R}^{n+1}$, then

$$G = \frac{1}{r} \Rightarrow L = \frac{1}{r} \Rightarrow R(X,Y)Z = \frac{1}{r^2}(\langle Y, Z\rangle X - \langle X, Z\rangle Y) \Rightarrow K(X,Y) = \frac{1}{r^2}$$

Remark. $(M, \langle\ ,\ \rangle)$ is any Riemannian manifold. Consider $(M, \underbrace{\lambda\langle\ ,\ \rangle}_{\langle\ ,\ \rangle_\lambda})$ for $\lambda > 0$. Then $K(X,Y)_{\langle\ ,\ \rangle_\lambda} = \frac{1}{\lambda} K(X,Y)_{\langle\ ,\ \rangle}$.

Chapter 14

The Euler Class

If $E \xrightarrow{\pi} M$ is a vector bundle of rank 1, $\langle\ ,\ \rangle$ a metric on E, then every metric compatible connection ∇ is flat.

Now take E of rank $k = 2$. ∇ is connection on E compatible with a metric $\langle\ ,\ \rangle$. Let s_1, s_2 be a local orthogonal frame with respect to $\langle\ ,\ \rangle$.

$$\nabla s_i = \sum_{j=1}^{2} \omega_{ij} \otimes s_j \text{ with } \omega_{ij} \text{ skew-symmetric } \begin{pmatrix} 0 & -\omega_{12} \\ -\omega_{21} & 0 \end{pmatrix}$$

Then

$$\Omega_{ij} = d\omega_{ij} - \sum_{l=1}^{2} \omega_{il} \wedge \omega_{lj}$$

Ω_{ij} is also skew-symmetric.

$$\Omega_{12} = d\omega_{12} - \sum_{l=1}^{2} \omega_{1l} \wedge \omega_{l2} = d\omega_{12} \quad \Rightarrow \quad d\Omega_{12} = 0$$

We assume now that E is oriented and s_1, s_2 are positive with respect to this orientation. Let s_1', s_2' be another local orthogonal frame, which is also positive oriented.

$$s_i' = \sum_{j=1}^{2} g_{ij} s_j$$

s_i, s_i' are defined on $E\Big|_U$, $g_{ij} \in \mathcal{C}^\infty(U)$. $g \in SO(2) = S^1$ at every point.

$$\begin{aligned} g &= \begin{pmatrix} \cos(f(x)) & -\sin(f(x)) \\ \sin(f(x)) & \cos(f(x)) \end{pmatrix} \Omega' \\ &= g\Omega g^{-1} \\ &= \begin{pmatrix} \cos(f(x)) & -\sin(f(x)) \\ \sin(f(x)) & \cos(f(x)) \end{pmatrix} \begin{pmatrix} 0 & \Omega_{12} \\ -\Omega_{21} & 0 \end{pmatrix} \begin{pmatrix} \cos(f(x)) & \sin(f(x)) \\ -\sin(f(x)) & \cos(f(x)) \end{pmatrix} \\ &= \begin{pmatrix} 0 & \Omega_{12} \\ -\Omega_{12}' & 0 \end{pmatrix} = \Omega \end{aligned}$$

The last equality is because $SO(2)$ is abelian. This shows that Ω and therefore Ω_{12} is independent of the choice of **oriented orthogonal frames** s_1**,** s_2.

Ω_{12} is a globally well-defined closed 2-form.

Definition 14.1. $e(E) := -\frac{1}{2\pi}[\Omega_{12}] \in H^2_{dR}(M)$.

Proposition 14.1.

(1) $e(\overline{E}) = -e(E)$, $\overline{E}$ is the vector bundle with opposite orientation.

(2) If E admits a section s, which is nowhere zero, then $e(E) = 0$ [without loss of generality $\langle s, s\rangle = 1$. Then $0 = 2\langle s, \nabla s\rangle$, so $\langle s, \nabla s\rangle = 0$. Take $s_1 = s$. There is a unique s_2, s.t. s_1, s_2 are orthogonal and oriented. Globally $\Omega_{12} = d\omega_{12}$, so $[\Omega] = 0 \in H^2_{dR}(M)$.]

(3) The Euler class is independent of the choice of ∇ (compatible with a fixed $\langle\ ,\ \rangle$). [Let ∇^0, ∇^1 be two different connections compatible with $\langle\ ,\ \rangle$. Then $\nabla^1 - \nabla^0 = A \in \Omega^1(\overbrace{\text{Skew-End}(E)}^{\text{rank}=1})$, with respect to a local orthogonal frame s_1, s_2:

$$\begin{pmatrix} 0 & \omega^1_{12} \\ -\omega^1_{12} & 0 \end{pmatrix} - \begin{pmatrix} 0 & \omega^0_{12} \\ -\omega^0_{12} & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$$

$a \in \Omega^1(M)$ is a globally well-defined 1-form, where a has trivial gauge transformation.

$$\omega^1_{12} = \omega^0_{12} + a \;\Rightarrow\; \Omega^1_{12} = \Omega^0_{12} + da \;\Rightarrow\; [\Omega^1_{12}] = [\Omega^0_{12}] \in H^2_{dR}(M)]$$

E of rank k is trivial if and only if $\exists s_1, \ldots, s_k$ sections, which are everywhere linear independent. E oriented of rank k is trivial if and only if $\exists s_1, \ldots, s_{k-1}$ which are everywhere linear independent.

(4) $e(E)$ is independent of the choice of metric.

[Sketch of proof: $E \times [0,1] \xrightarrow{\pi\in\text{id}} M \times [0,1]$ as a $\mathbb{E}$ vector bundle on $M \times [0,1]$. On $\mathbb{E}_{(x,t)}$, we consider the metric $(1-t)\langle -, -\rangle^0_x + t\langle -, -\rangle^1_x = \langle\langle -, -\rangle\rangle_{x,t}$. This is a metric on $\mathbb{E}$, which restricts to $\mathbb{E}\Big|_{M\times\{0\}} = E$ as $\langle\ ,\ \rangle^0$ and to $\mathbb{E}\Big|_{M\times\{1\}}$ as $\langle\ ,\ \rangle^1$. Let $\mathbb{\nabla}$ be a connection on $\mathbb{E}$ compatible with $\langle\langle\ ,\ \rangle\rangle$. From its curvature, we determine $e(\mathbb{E}) \in H^2_{dR}(M \times [0,1])$. Let $i_0, i_1 : M \hookrightarrow M \times [0,1]$, where $i_0(x) = (x, 0)$ and $i_1(x) = (x, 1)$. Then $e(E, \langle\ ,\ \rangle^0) = i_0^* e(\mathbb{E}, \langle\langle\ ,\ \rangle\rangle)$. Similarly, $e(E, \langle\ ,\ \rangle^1) = i_1^* e(\mathbb{E}, \langle\langle\ ,\ \rangle\rangle)$. $\Rightarrow$ $e(E, \langle\ ,\ \rangle^0) = e(E, \langle\ ,\ \rangle^1)$, because $i_0^* = \text{Id} = i_1^*$. By Poincaré lemma, i_0^* and i_1^* are homotopic maps induce the same H_{dR}.]

Example 14.1. $M = S^2$. Take two copies of $\mathbb{R}^2 \times \mathbb{R}^2$. With the standard $\langle\ ,\ \rangle$ on the second factor. And standard flat connection compatible with $\langle\ ,\ \rangle$. Take $\psi : (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2 \to (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2$ where $g : \mathbb{R}^2 \setminus \{0\} \to SO(2)$. $X_1 =$

$$(x, v) \mapsto \left(-\frac{x}{\|x\|^2}, g(x)v\right)$$

$\mathbb{R}^2 \times \mathbb{R}^2$, $X_2 = \mathbb{R}^2 \times \mathbb{R}^2$ are identified via ψ to get an oriented rank 2 vector bundle $E \to S^2$, with a metric.

Let ∇^0 be the given flat connection on $E\Big|_{S^2\setminus\{N\}}$, coming from X_1.

Let ∇^1 be the given flat connection on $E\Big|_{S^2\setminus\{S\}}$, coming from X_2.

Choose a smooth partition of unity ρ, $1-\rho$ subordinate to the covering of S^2 by $S^2 \setminus \{N\}$ and $S^2 \setminus \{S\}$. Write $S^2 \setminus \{N, S\} = S^1 \times \mathbb{R}$.

$$\begin{aligned} \rho : S^1 \times \mathbb{R} &\to \mathbb{R} \\ (\varphi, t) &\mapsto \rho(t) \end{aligned}$$

ρ extends to a smooth function on S^2. Define $\nabla = \rho\nabla^1 + (1-\rho)\nabla^0$. This is a metric connection on $E \to S^2$. Over $S^2 \setminus \{N, S\}$, consider the frame which is parallel for ∇^0 given by the standard basis for $\mathbb{R}^2$. With respect to this frame the connection matrix for ∇ is that for ∇^1 scaled by ρ.

Let s_1', s_2' be the parallel frame for ∇^1 coming from X_2. In this frame, ∇^1 has zero connection matrix.

$$\begin{aligned} 0 = \omega' &= dgg^{-1} + g\omega g^{-1} \\ \Rightarrow \qquad g\omega g^{-1} &= -dgg^{-1} \\ \Rightarrow \qquad \omega &= -g^{-1}dg \end{aligned}$$

$$\omega_{12} = -\sum_{i=1}^{2} g^{1i} dg_{i2}$$

Take $g : S^1 \times \mathbb{R} \to SO(2)$, we could also take $g = \mathrm{e}^{\mathrm{i}n\varphi}$.

$$(\varphi, t) \mapsto \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

$$\Rightarrow \qquad \omega_{12} = -g^{11}dg_{12} - g^{12}dg_{22} = d\varphi$$

In the frame s_1, s_2, ∇ is represented by $\begin{pmatrix} 0 & \rho d\varphi \\ -\rho d\varphi & 0 \end{pmatrix}$

$$\Omega_{12} = d(\rho d\varphi) = d\rho \wedge \underbrace{d\varphi}_{\text{Not really exact on } S^1} = \frac{d\rho}{dt} dt \wedge d\varphi$$

$$\int_{S^2} \Omega_{12} = \int_{S^1\times\mathbb{R}} \Omega_{12} = -\left(\int_{-\infty}^{+\infty} dt \frac{d\rho}{dt}\right)\left(\int_{S^1} d\varphi\right) = -(\rho(\infty) - \rho(-\infty)) \cdot 2\pi = -2\pi \neq 0$$

with $g = \mathrm{e}^{\mathrm{i}n\varphi}$: $\int_{S^2} \Omega_{12} = -2\pi n$, g is called clutching map. We can do this for general S^n.

If E, F are oriented preserving isomorphism, then $e(E) = e(F)$.

If M is oriented, n-dimensional, then

$$\int_M : H^n_c(M) \to \mathbb{R}$$

is well defined and surjective.

M is an oriented 2-dimensional manifold, compact without boundary, then

$$\int_M : H^2_{dR}(M) \to \mathbb{R}$$

If M is connected, this is an isomorphism.

Definition 14.2. $\chi(E) := \int_M e(E)$ is the Euler number of E.

Let M be oriented 2-dimensional manifold, and $\langle\ ,\ \rangle$ a Riemannian metric.

How do we determine $e(TM)$?

Let X_1, X_2 be a local orthogonal frame for $(TM, \langle\ ,\ \rangle)$, s.t. (X_1, X_2) is positive oriented.

$$K(T_pM) = \langle R(X_1,X_2)X_2, X_1\rangle = \langle \nabla_{X_1}\nabla_{X_2}X_2 - \nabla_{X_2}\nabla_{X_1}X_2 - \nabla_{[X_1,X_2]}X_2, X_1\rangle$$

where ∇_{X_i}, $i = 1, 2$ is the Levi-Civita connection.

$$\begin{array}{l} \nabla X_1 = \omega_{12} \otimes X_2 \\ \nabla X_2 = -\omega_{12} \otimes X_1 \end{array} \Rightarrow \begin{array}{r} \nabla_{X_2}X_2 = -\omega_{12}(X_2)X_1 \\ \nabla_{X_1}X_1 = -\omega_{12}(X_1)X_1 \\ \nabla_{[X_1,X_2]}X_2 = -\omega_{12}([X_1,X_2])X_1 \end{array}$$

Therefore,

$$\begin{aligned} K(T_pM) &= \langle \nabla_{X_1}(-\omega_{12}(X_2)X_1) - \nabla_{X_2}(-\omega_{12}(X_1)X_1) + \omega_{12}\langle([X_1,X_2])X_1, X_1\rangle \\ &= \langle -L_{X_1}\omega_{12}(X_2)\cdot X_1 - \cancel{\omega_{12}(X_2)\nabla_{X_1}X_1} + L_{X_2}\omega_{12}(X_1)\cdot X_1 \\ &+ \cancel{\omega_{12}(X_1)\nabla_{X_2}X_1} + \omega_{12}([X_1,X_2])X_1, X_1\rangle \\ &= -L_{X_1}\omega_{12}(X_2) + L_{X_2}\omega_{12}(X_1) + \omega_{12}([X_1,X_2]) \\ &= -(d\omega_{12})(X_1,X_2) \\ &= -\Omega_{12}(X_1,X_2) \end{aligned}$$

[M n-dimensional oriented, $\langle\ ,\ \rangle$ on $TM \Rightarrow \exists!\ dvol \in \Omega^n(M)$ with $dvol(X_1,\ldots,X_n) = 1$ for any oriented orthonormal basis $X_1,\ldots,X_n$ of T_pM. $dvol = X_1^* \wedge \cdots \wedge X_n^*$.]

Theorem 14.2 (Gauss Bonnet Theorem)**.** On an oriented 2-dimensional manifold with a metric, the equation $K(T_pM) = -\Omega_{12}(X_1,X_2)$ is equivalent to $\Omega_{12} = -K \cdot dvol$. The Euler number of $TM \xrightarrow{\pi} M$ is

$$\boxed{\chi(TM) = -\frac{1}{2\pi}\int_M \Omega_{12} = \frac{1}{2\pi}\int_M K\cdot dvol}$$

where $\chi(TM)$ is the Euler character of M.

Example 14.2. $M = S^2(R)$ is the 2-sphere of radius R in $\mathbb{R}^3$.

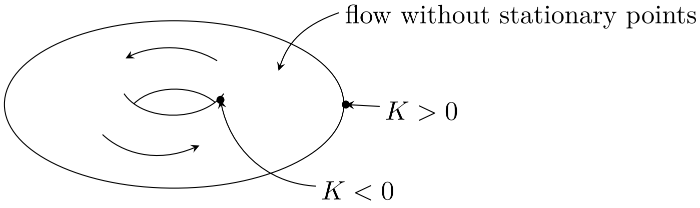
$$\chi(TS^2) = \frac{1}{2\pi} \int\limits_{S^2(R)} K \cdot dvol = \frac{1}{2\pi R^2} \cdot vol(S^2(R)) = \frac{4\pi R^2}{2\pi R^2} = 2$$

Example 14.3. Suppose the 2-manifold M admits a vector field without zeroes. Then

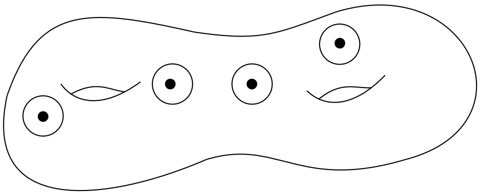
$$\chi(TM) = 0 = \frac{1}{2\pi} \int\limits_{M} K \cdot dvol$$

Corollary 14.3 (Hedgehog/Hairy Ball Theorem)**.** S^2 does not admit a vector field without zeroes.

Example 14.4. $M = T^2$.



M an oriented 2-dimensional manifold, $X \in \mathfrak{X}(M)$ a vector field with isolated zeroes.



M connected, compact $\Rightarrow$ X has finitely many zeroes $p_1, \ldots, p_k$.

Choose disjoint open neighborhoods $U_1, \ldots, U_k$ of $p_1, \ldots, p_k$, with each U_i diffeomorphic to a disc of radius 2 in $\mathbb{R}^2$ and $V_i = D(1)$ in.

We equip M with a Riemannian metric, which restricts to each U_i as the flat metric of $U_i \subset \mathbb{R}^2$.

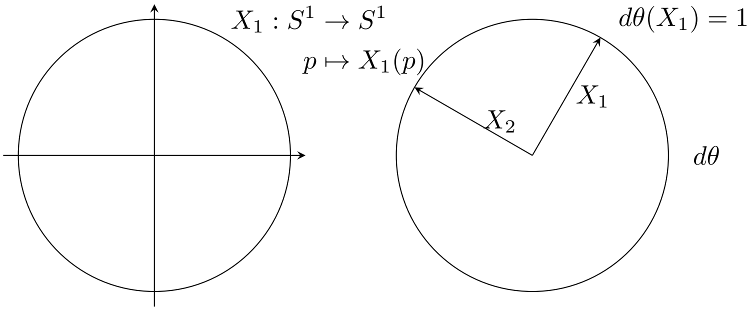
On $M \setminus \{p_1, \ldots, p_k\}$, define $X_1 = \dfrac{X}{\|X\|}$ with respect to our metric.

Complete X_1 to an oriented orthonormal basis $\{X_1, X_2\}$ on $M \backslash \{p_1, \dots, p_k\}$.

$$\begin{aligned}
\chi(TM) &= \frac{1}{2\pi} \int\limits_M K \cdot dvol \\
&\overset{\text{flat around poles}}{=\!=\!=} \frac{1}{2\pi} \int\limits_{M \backslash (\bigcup V_i)} K \cdot dvol \\
&= -\frac{1}{2\pi} \int\limits_{M \backslash (\bigcup V_i)} \Omega_{12} \\
&\overset{\text{Stokes}}{=\!=\!=} -\frac{1}{2\pi} \int\limits_{\partial (M \backslash (\cup V_i))} \omega_{12} \\
&= \frac{1}{2\pi} \sum_{i=1}^{n} \int\limits_{\partial V_i} \omega_{12}
\end{aligned}$$

Claim 14.4. $X_1^* d\theta = \omega_{12}$.

Proof. Notice that



$$(X_1^* d\theta)(Y) = d\theta((DX_1)Y) \overset{\text{flat connection}}{=\!=\!=} d\theta(\nabla_Y X_1) = d\theta(\omega_{12}(Y) X_2) = \omega_{12} Y$$

□

$$\Rightarrow \quad \chi(TM) = \frac{1}{2\pi} \sum_{i=1}^{n} \int\limits_{\partial V_i} X_1^* d\theta = \frac{1}{2\pi} \sum_{i=1}^{n} \deg \left(X_1 \Big|_{\partial V_i} \right) \int\limits_{S^1} d\theta = \sum_{i=1}^{n} \underbrace{\deg \left(X_1 \Big|_{\partial V_i} \right)}_{= \mathrm{Index}(X_1, p_i)}$$

This is called the Poincaré–Hopf Theorem.