Category Theory Problem Sheets

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Problem Sheet 1

Foundations of Categories & Functors

Exercise 1.1. Find several examples of categories and functors from some of the other lectures which you are attending, and show that these satisfy the definition of a category (resp. a functor).

Exercise 1.2. Show that the identity morphisms in a category are uniquely determined. That is, for each object c of a category C, show that there exists a unique element $\mathrm{id}_c \in \mathrm{Hom}(c, c)$ such that for all objects d of C and for all $f \in \mathrm{Hom}(c, d)$ (resp. all $f \in \mathrm{Hom}(d, c)$) we have $f \circ \mathrm{id}_c = f$ (resp. $\mathrm{id}_c \circ f = f$).

Problem Sheet 2

Functoriality & Duality

Exercise 2.1. Find several examples of functors from some of the other lectures which you are attending, and show that these satisfy the definition of a functor.

Exercise 2.2. Show that any functor sends isomorphisms to isomorphisms.

Exercise 2.3. Given categories C and D, what is the difference between a functor $C^{\text{op}} \to D$ and a functor $C \to D^{\text{op}}$? What is the difference between a functor $C \to D$ and a functor $C^{\text{op}} \to D^{\text{op}}$?

Problem Sheet 3

Natural Transformations & Group Actions

Exercise 3.1. Find several examples of natural transformations from some of the other lectures which you are attending, and show that these satisfy the definition of a natural transformation.

Exercise 3.2. Let $F, G : C \to D$ be two functors and $\alpha : F \to G$ be a natural transformation. Show that α is an isomorphism of functors, i.e. admits a two-sided inverse natural transformation, if and only if for all $c \in C$, the morphism $\alpha_c : Fc \to Gc$ in D is an isomorphism.

Exercise 3.3. Let G and H be groups. Give concrete descriptions of the following objects:

- Functors $BG \to BH$.
- Functors $BG \to \mathbf{Set}$.
- Functors $BG \to C$ for an arbitrary category C.
- Natural transformations between the above functors.

Problem Sheet 4

Properties of Functors & Equivalence

Exercise 4.1. Consider the following properties of functors: faithful, full, essentially surjective. For each combination of these properties, find a functor having exactly these properties.

Exercise 4.2. Show that the composition of two equivalences of categories is again an equivalence.

Exercise 4.3. Let C be the category of finitely generated abelian groups.

(1) Show that for each $n \in \mathbb{Z}$, there is a natural transformation $\alpha_n : \mathrm{id}_C \to \mathrm{id}_C$ which for every $A \in C$ is given by the homomorphism $A \to A, a \mapsto na$.

(2) Show that every natural transformation $\mathrm{id}_C \to \mathrm{id}_C$ is equal to α_n for some $n \in \mathbb{Z}$.

(3) For $A \in C$, let $TA \subset A$ be the torsion subgroup of A. We consider the functor

$$\begin{aligned} G : C &\to C \\ A &\mapsto TA \oplus A/TA \end{aligned}$$

By the structure theorem for finitely generated abelian groups, for each $A \in C$ there exists an isomorphism

$$A \cong TA \oplus A/TA$$

Show that there is no way to choose such isomorphisms for all $A \in C$ in such a way that they form an isomorphism $\mathrm{id}_C \cong G$ of functors.

Problem Sheet 5

Universal Properties & (Co)Limits in Orders

Exercise 5.1. For some of the examples of universal properties which we have seen in the lecture, find (new) examples of categories in which objects with these universal properties always exist, and also examples of categories in which objects with these universal properties don't always exist.

Exercise 5.2. Consider the category associated to the preordered set $(\mathbb{Z}, \leqslant)$, that is the category whose objects are the elements of $\mathbb{Z}$ and whose morphisms are given by

$$\mathrm{Hom}(n,m) = \begin{cases} \{*\} & \text{if } n \leqslant m \\ \varnothing & \text{else} \end{cases}$$

with the only possible composition rule. Describe products and coproducts in this category.

Exercise 5.3. Is the category of sets equivalent to its opposite category?

Problem Sheet 6

Representable Functors & Emptiness

Exercise 6.1. Find some more examples of representable functors on your favorite categories.

Exercise 6.2. Let C be a category. Show that the functor $C \to \mathbf{Set}$ which sends every object to the empty set and every morphism to the unique map from the empty set to itself is not representable.

Exercise 6.3. Consider the categories $\mathbb{1}$ (resp. $\mathbb{2}$) with exactly one (resp. two) objects and only the identity morphisms, as well as the category $\mathbf{Cat}$ of small categories. Describe the functors $\mathbf{Cat} \to \mathbf{Set}$ represented by $\mathbb{1}$ and $\mathbb{2}$.

Problem Sheet 7

Yoneda's Lens: Embeddings and Transformations

Exercise 7.1. Use the Yoneda lemma to compute the endomorphisms of some representable functors.

Exercise 7.2. Let G be a group. By applying the Yoneda lemma to the category BG, prove the theorem of Cayley that G admits an embedding into the symmetric group of its underlying set.

Exercise 7.3. Use the Yoneda lemma to answer the following question: Does there exist a non-identity natural transformation from the identity functor on the category of topological spaces to itself?

Problem Sheet 8

(Co)Limits: Existence and Uniqueness

Exercise 8.1. For some of your favorite categories from some of the other lectures which you are attending, determine which (co)limits exist in these categories.

Exercise 8.2. Explicitly describe small limits and colimits in the category of sets.

Exercise 8.3. Let I be a category with an initial object i_0.

(1) Show that for any functor $F : I \to C$, the limit $\lim_I F$ exists and describe this limit explicitly.

(2) Show that for any $i \in I$, the coproduct $i \coprod i_0$ exists and describe it explicitly.

Exercise 8.4. Find an example of non-isomorphic objects c and d of some category C, such that for all objects $e \in C$ there exists a bijection $\mathrm{Hom}(c, e) \cong \mathrm{Hom}(d, e)$ of Hom-sets.

Problem Sheet 9

Groups, (Co)Limits, and Functor Completeness

Exercise 9.1. For a group G and a functor $F : BG \to \mathbf{Set}$, describe $\lim_{BG} F$ and $\operatorname{colim}_{BG} F$.

Exercise 9.2. For a small category C, show that $\mathrm{Fun}(C, \mathbf{Set})$ is complete and cocomplete and describe small (co)limits in this category.

Exercise 9.3. Let C be a locally small category and $c \in C$. Show that the functor $\mathrm{Hom}(c, -)$ preserves all limits which exist in C.

Exercise 9.4. Show that every group can be written as a colimit of a diagram consisting of finitely generated groups.

Problem Sheet 10

Adjunctions: Constructions and Obstructions

Exercise 10.1. Find examples of adjunctions from some of the other lectures which you are attending.

Exercise 10.2. Given categories C_1, C_2, C_3 and adjunctions $(F_1 : C_1 \to C_2, G_1 : C_2 \to C_1)$ and $(F_2 : C_2 \to C_3, G_2 : C_3 \to C_2)$ construct a natural adjunction $(F_2 \circ F_1, G_1 \circ G_2)$.

Exercise 10.3. Does the forgetful functor $\mathbf{AbGrp} \to \mathbf{Grp}$ admit a right adjoint?

Exercise 10.4. Find right and left adjoints to the functor from the category of small categories to the category of sets which sends a small category to its set of objects and a functor to the induced map on the sets of objects.

Problem Sheet 11

Adjoint Theorems & Representability

Exercise 11.1. Show that the forgetful functor from the category of fields to the category of rings admits neither a left nor a right adjoint.

Exercise 11.2. Use the adjoint functor theorem to prove the existence of free groups.

Exercise 11.3. Let C be a locally small category which admits all small coproducts. Show that a functor $F : C \to \mathbf{Set}$ is representable if and only if it admits a left adjoint.

Problem Sheet 12

Monads, Algebras, and Adjunction Diagrams

Exercise 12.1. Consider functors $F : C \to D$ and $G : D \to C$.

(1) Consider the natural transformations $\eta : \mathrm{id}_C \to GF$ and $\epsilon : FG \to \mathrm{id}_D$ associated to an adjunction between F and G. Show that the following diagrams of natural transformations commute:

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "\mathrm{id}_F"'] & FGF \arrow[d, "\epsilon F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\eta G"] \arrow[dr, "\mathrm{id}_G"'] & GFG \arrow[d, "G\epsilon"] \\ & G \end{tikzcd}$$

(2) Inversely, from such natural transformations η and ϵ for which the above triangles commute, construct an adjunction between F and G.

Exercise 12.2. Consider the forgetful functor from the category of partially ordered sets to the category of sets. Show that this functor has a left adjoint and describe the associated monad T: $\mathbf{Set} \to \mathbf{Set}$ as well as the category $\mathbf{Set}^T$ of T-algebras.

Exercise 12.3. Let T be a monad on a category C. Show that for any index category I, if I-indexed limits exist in C, then they exist in C^T.