Category Theory

Paul Ziegler

Preface

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Chapter 1

Categories

1.1 Introduction

This brief introduction is taken from Awodey's book.

What is category theory? As a first approximation, one could say that category theory is the mathematical study of (abstract) algebras of functions.

The historical development of the subject has been, very roughly, as follows:

- 1945: Eilenberg and Mac Lane's "General theory of natural equivalences" was the original paper, in which the theory was first formulated.
- Late 1940s: The main applications were originally in the fields of algebraic topology, particularly homology theory, and abstract algebra.
- 1950s: Grothendieck et al. began using category theory with great success in algebraic geometry.
- 1960s: Lawvere and others began applying categories to logic, revealing some deep and surprising connections.
- 1970s: Applications were already appearing in computer science, linguistics, cognitive science, philosophy, and many other areas.

One very striking thing about the field is that it has such wide-ranging applications. In fact, it turns out to be a kind of universal mathematical language like Set Theory. As a result of these various applications, category theory also tends to reveal certain connections between different fields like Logic and Geometry.

In fact, just as the idea of a topological space arose in connection with continuous functions, so also the notion of a category arose in order to define that of a functor, at least according to one of the inventors. The notion of a functor arose - so the story goes on - in order to define natural transformations. One might as well continue that natural transformations serve to define adjoints, so we have the following succession:

$$\text{category} \rightsquigarrow \text{functor} \rightsquigarrow \text{natural transformation} \rightsquigarrow \text{adjunction}.$$

Before getting down to business, let us ask why it should be that category theory has such far-reaching applications. Well, we said that it is the abstract theory of functions, so the answer is simply this:

Functions are everywhere!

And everywhere that functions are, there are categories. Indeed, the subject might better have been called abstract function theory, or, perhaps even better: archery.

1.2 Definition

Definition 1.1. A category $\mathcal{C}$ consists of the following data:

(1) **objects**: $A, B, C, \ldots$ (not necessarily sets). The class of objects of $\mathcal{C}$ is denoted by $\mathrm{Ob}(\mathcal{C})$.

(2) **arrows** (or **morphisms**): $f, g, h, \ldots$ (not necessarily functions). The class of arrows of $\mathcal{C}$ is denoted by $\mathrm{Mor}(\mathcal{C})$. The class of arrows between two objects A, B of $\mathcal{C}$ is denoted by $\mathrm{Hom}_{\mathcal{C}}(A, B)$ or $\mathrm{Mor}_{\mathcal{C}}(A, B)$.

(3) For every arrow f there are objects $\mathrm{dom}(f)$ and $\mathrm{cod}(f)$ called the **domain** and the **codomain** of f respectively. We write $f : A \to B$, where $A = \mathrm{dom}(f)$ and $B = \mathrm{cod}(f)$.

(4) For every arrows $f : A \to B$ and $g : B \to C$ there is an arrow denoted $g \circ f : A \to C$ and called the **composite** of f and g.

(5) For every object A there is an arrow denoted $1_A : A \to A$ and called the **identity arrow**.

which satisfy the following axioms:

(i) Associativity:

$$(h \circ g) \circ f = h \circ (g \circ f), \quad \forall f : A \to B, g : B \to C, h : C \to D$$

(ii) Unit:

$$f \circ 1_A = f = 1_B \circ f, \quad \forall f : A \to B$$

If A, B are objects and $f : A \to B$ is an arrow in a category $\mathcal{C}$, then sometimes we simply denote $A, B \in \mathcal{C}$ and $f \in \mathcal{C}$ instead of $A, B \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.

1.3 Examples

1.3.1 The Category Set

- objects: sets
- arrows: functions
- composition: composition of functions
- identity arrow: identity function

There are variations of this category obtained by restricting the sets and/or the functions, such as: the category $\mathbf{Set}_{fin}$ of finite sets and functions, the category of sets and injective functions etc.

1.3.2 Categories of Structured Sets

(1) The category of groups and group homomorphisms, denoted by **Grp**. The category of abelian groups and group homomorphisms, denoted by **Ab**.

(2) The category of monoids and monoid homomorphisms, denoted by **Mon**.

(3) The category of unitary rings and unitary ring homomorphisms, denoted by **Ring**. The category of commutative unitary rings and unitary ring homomorphisms, denoted by **CRing**.

(4) The category of rings and ring homomorphisms, denoted by **Rng**. The category of commutative rings and ring homomorphisms, denoted by **CRng**.

(5) The category of fields and field homomorphisms, denoted by **Field**.

(6) The category of left vector spaces over a field K and K-linear maps, denoted by $\mathbf{Vect}(K)$. The category of left modules over a unitary ring R and R-module homomorphisms, denoted by $\mathbf{Mod}(R)$.

(7) The category of graphs and graph homomorphisms, denoted by **Graph**.

(8) The category of topological spaces and continuous maps, denoted by **Top**.

(9) The category of real Banach spaces and linear contractions, denoted by **Ban**.

(10) The category of differentiable (smooth) manifolds and differentiable (smooth) mappings, denoted by **Man**.

(11) The category of preordered sets and monotone mappings, denoted by **Preord**. The category of posets (partially ordered sets) and monotone mappings, denoted by **Pos**.

Remark. All the above examples are concrete categories, which roughly speaking means that the objects are some sets and the arrows are some functions.

1.3.3 The Category Rel

- objects: sets
- arrows: relations $r = (A, B, R)$, where $R \subseteq A \times B$
- composition: composition of relations, defined for relations $r = (A, B, R)$ and $s = (B, C, S)$ as $s \circ r = (A, C, S \circ R)$, where

$$S \circ R = \{(a, c) \in A \times C \mid \exists b \in B \text{ such that } (a, b) \in R \textit{ and } (b, c) \in S\}$$

- identity arrow: for every set A, the identity arrow is the equality relation $\delta_A = (A, A, \Delta_A)$, where $\Delta_A = \{(a, a) \mid a \in A\}$.

1.3.4 Another Category of Finite Sets: $\mathbf{Mat}(K)$

- Objects: finite sets (or simply natural numbers)
- arrows: for every finite sets A with $|A| = m \in \mathbb{N}$ and B with $|B| = n \in \mathbb{N}$, define an arrow $A \to B$ to be a matrix in $M_{m,n}(K)$ (where K is a fixed field).
- composition: multiplication of matrices
- identity arrow: identity matrix

1.3.5 Poset Categories

Given a poset $(P, \leqslant)$, we may construct an associated category called a **poset category**:

- Objects: the elements of P
- arrows: we say that there is an arrow between $a, b \in P$, and we write $a \to b$, if and only if $a \leqslant b$.
- composition: composition of arrows in the sense that $a \to b \to c$ if and only if $a \leqslant b \leqslant c$
- identity arrow: we have $a \to a$ for every object $a \in P$.

1.3.6 Monoid Categories

Given a monoid $(M, \cdot)$, we may construct an associated category:

- objects: the single object M
- arrows: the elements of M
- composition: the multiplication of the elements of M
- identity arrow: the identity element from the monoid

1.3.7 Finite Categories

(1) The category $\mathbf{0}$: it has no objects and no arrows.

(2) The category $\mathbf{1}$:

$$*$$

which has a single object $*$, the identity arrow as the single arrow and the composition is given by iteration of the identity arrow.

(3) The category $\mathbf{2}$:

$$* \to *$$

which has two objects, the identity arrows and one non-identity arrow between the two objects and the composition is given by two succesive arrows.

(4) The category **3**:

$$\begin{tikzcd} * \arrow[r, "f"] \arrow[dr, "g\circ f"'] & * \arrow[d, "g"] \\ & * \end{tikzcd}$$

which has 3 objects, the identity arrows and the 3 depicted non-identity arrows between the objects and the composition is given by two successive arrows.

1.3.8 The Category Cat

Definition 1.2. A **covariant functor** (or **simply functor**) between two categories $\mathcal{C}$ and $\mathcal{D}$ is a mapping of objects of $\mathcal{C}$ to objects of $\mathcal{D}$ and of arrows of $\mathcal{C}$ to arrows of $\mathcal{D}$, denoted by

$$F : \mathcal{C} \to \mathcal{D}$$

satisfying the axioms:

(i) For every $f : A \to B$ in $\mathcal{C}$, we have $F(f) : F(A) \to F(B)$ in $\mathcal{D}$.

(ii) For every object A of $\mathcal{C}$, we have $F(1_A) = 1_{F(A)}$.

(iii) For every composable pair of arrows $f : A \to B$ and $g : B \to C$ in $\mathcal{C}$, we have

$$F(g \circ f) = F(g) \circ F(f)$$

hence the commutativity of the following left diagram implies the commutativity of the right diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[dr, "F(g\circ f)"'] & F(B) \arrow[d, "F(g)"] \\ & F(C) \end{tikzcd}$$

For functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$, one defines the composite functor $G \circ F : \mathcal{C} \to \mathcal{E}$ by

$$\begin{aligned} (G \circ F)(C) &= G(F(C)), && \text{for every object } C \text{ of } \mathcal{C} \\ (G \circ F)(f) &= G(F(f)), && \text{for every arrow } f : A \to B \text{ of } \mathcal{C} \end{aligned}$$

The category **Cat**:

- objects: small categories (that is, categories $\mathcal{C}$ such that $\mathrm{Ob}(\mathcal{C})$ and $\mathrm{Mor}(\mathcal{C})$ are sets)
- arrows: covariant functors
- composition: composition of functors
- identity arrow: identity functor $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ for every category $\mathcal{C}$, defined by the identity on objects and on arrows.

1.3.9 A Category form Logic

Given a deductive system of logic, we may construct an associated category of proofs:

- objects: formulas $\varphi, \psi, \dots$
- arrows: implications $\varphi \to \psi$
- composition: succesive implications $\varphi \to \psi \to \Delta$
- identity arrow: each formula implies itself

1.3.10 A Category from Computer Science

Given a functional programming language L, we may construct an associated category:

- objects: data types of L
- arrows: computable functions on L ("processes")
- composition: succesive computable functions $X \to Y \to Z$, where the output of the first arrow is the input of the second arrow
- identity arrow: "do nothing" procedure

1.3.11 A Category from Physics

- objects: physical system $A, B, C, \dots$
- arrows: physical processes which take a physical system of type of A into a physical system of type B
- composition: sequential composition of physical processes
- identity arrow: the physical process leaving the physical system invariant

1.4 Isomorphisms

Definition 1.3. In any category $\mathcal{C}$, an arrow $f : A \to B$ is called an **isomorphism** if there is an arrow $g : B \to A$ such that

$$g \circ f = 1_A \quad \text{and} \quad f \circ g = 1_B$$

Since inverses are unique, we write $g = f^{-1}$.

We say that A is **isomorphic** to B, written $A \cong B$ if there exists an isomorphism between them.

We recall the following famous theorem from Group Theory.

Theorem 1.1 (Cayley)**.** Every group is isomorphic to a subgroup of a symmetric group.

Proof. Let $(G, \cdot)$ be a group and consider the symmetric group

$$S_G = \{g : G \to G \mid g \text{ is bijective}\}.$$

For every $a \in G$, define

$$t_a : G \to G \text{ by } t_a(x) = ax, \qquad \forall x \in G.$$

One proves that $t_a \in S_G$, that is t_a is bijective. We may now define

$$f : G \to S_G \text{ by } f(a) = t_a, \forall a \in G$$

One shows that f is an injective homomorphism. Then $\operatorname{Ker} f = \{1\}$.

By the first isomorphism theorem, it follows that

$$G \cong G/\{1\} \cong G/\operatorname{Ker} f \cong \operatorname{Im} f.$$

But $\operatorname{Im} f$ is a subgroup of S_G, so that we are done. Note that $\operatorname{Im} f$ is sometimes called the Cayley representation of G. □

Remark. Note the two different levels of isomorphisms that occur in the proof of Cayley's theorem. There are bijective functions $g : G \to G$, which are isomorphisms in **Set**, and there is the isomorphism between G and $\operatorname{Im} f$ in **Grp**. Cayley's theorem says that any abstract group can be represented as a "concrete" one, that is, a subgroup of a symmetric group.

We may give the following category-theoretic analogue.

Theorem 1.2. Every category $\mathcal{C}$ with a set of arrows is isomorphic to one in which the objects are sets and the arrows are functions.

Proof. Define the Cayley representation $\overline{\mathcal{C}}$ of $\mathcal{C}$, that is, the category corresponding to $\mathcal{C}$ via the isomorphism, to be the following concrete category:

- objects: sets of the form $\bar{C} = \{f \in \mathcal{C} \mid \operatorname{cod}(f) = C\}$ for objects C of $\mathcal{C}$
- arrows: functions $\bar{g} : \bar{C} \to \bar{D}$ for arrows $g : C \to D$ in $\mathcal{C}$, defined by $\bar{g}(f) = g \circ f$ for every $f : X \to C$ in $\bar{C}$.

One shows the required properties. □

1.5 Constructions on Categories

1.5.1 Product Category

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. The **product category** $\mathcal{C} \times \mathcal{D}$ is defined as follows:

- objects: pairs (C, D) for some objects $C \in \mathcal{C}$ and $D \in \mathcal{D}$
- arrows: pairs (f, g) for some arrows $f \in \mathcal{C}$ and $g \in \mathcal{D}$
- composition: for every composable arrows $(f, g), (f', g') \in \mathcal{C} \times \mathcal{D}$, their composite is defined as

$$(f, g) \circ (f', g') = (f \circ f', g \circ g')$$

- identity arrow: for every object $(C, D) \in \mathcal{C} \times \mathcal{D}$, the identity arrow is $1_{(C,D)} = (1_C, 1_D)$.

Clearly, the construction may be generalized for a finite number of categories.

1.5.2 Opposite Category

Let $\mathcal{C}$ be a category. The opposite category $\mathcal{C}^{\mathrm{op}}$ of $\mathcal{C}$ is defined as follows:

- objects: the objects of $\mathcal{C}$. We denote by C^* the object C of $\mathcal{C}$ viewed as an object of $\mathcal{C}^{\mathrm{op}}$.
- arrows: the arrows of the form $f^* : B^* \to A^*$ for some arrow $f : A \to B$ in $\mathcal{C}$.
- composition: for every arrows $f^* : B^* \to A^*$ and $g^* : C^* \to B^*$ in $\mathcal{C}^{\mathrm{op}}$, their composite is defined as

$$f^* \circ g^* = (g \circ f)^*$$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} A^* & B^* \arrow[l, "f^*"'] \\ & C^* \arrow[u, "g^*"'] \arrow[ul, "(g\circ f)^*"] \end{tikzcd}$$

- identity arrow: for every object $A^* \in \mathcal{C}^{\mathrm{op}}$, the identity arrow is $1_{A^*} = (1_A)^*$.

1.5.3 Arrow Category

Let $\mathcal{C}$ be a category. The **arrow category** $\mathcal{C}^{\to}$ of $\mathcal{C}$ is defined as follows:

- objects: the arrows of $\mathcal{C}$.
- arrows: an arrow $g = (g_1, g_2) : f \to f'$, where $f : A \to B$ and $f' : A' \to B'$ are arrows of $\mathcal{C}$, is a square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g_1"'] & B \arrow[d, "g_2"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd}$$

where $g_1 : A \to A'$ and $g_2 : B \to B'$ are arrows in $\mathcal{C}$, which is commutative in the sense that

$$f' \circ g_1 = g_2 \circ f$$

- composition: for every composable arrows (h_1, h_2) and (g_1, g_2) in $\mathcal{C}^{\to}$, their composite is defined as

$$(h_1, h_2) \circ (g_1, g_2) = (h_1 \circ g_1, h_2 \circ g_2)\,.$$

- identity arrow: for every object $f : A \to B$ in $\mathcal{C}^{\to}$, the identity arrow is $1_f = (1_A, 1_B)$.

1.6 Free Categories

Let A be a set, which will be called an "alphabet". Denote by A^* the set of all "words" of with "letters" from A, that is, strings of elements from A. We call A^* the **Kleene closure** of A. Denote by e the empty word. We immediately have the following result.

Theorem 1.3. Let A be a set. Consider on A^* the operation "$\cdot$" defined by concatenation. Then $(A^*, \cdot)$ is a monoid with identity element e, called the free monoid on A.

Theorem 1.4 (Universal Mapping Property of the Free Monoid)**.** With the above notation, there is an injective monoid homomorphism $i : A \hookrightarrow A^*$ with the property that for every monoid N and for every function $f : A \to N$, there is a unique monoid homomorphism $\bar{f} : A^* \to N$ such that $\bar{f} \circ i = f$, that is, the following diagram is commutative

$$
\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "f"'] & A^* \arrow[dl, dashed, "\bar{f}"] \\
N &
\end{tikzcd}
$$

Proof. Let $i : A \hookrightarrow A^*$ be the inclusion homomorphism, which is an injective monoid homomorphism. Define $\bar{f} : A^* \to N$ by

$$
\begin{aligned}
&\bar{f}(e) = e_N \\
&\bar{f}(w) = f(a_1) \cdots f(a_n), \quad \forall w = a_1 \cdots a_n \in A^*
\end{aligned}
$$

One checks that $\bar{f}$ is a monoid homomorphism and $\bar{f}(a) = f(a)$ for every $a \in A$, that is, $\bar{f} \circ i = f$.

For uniqueness, suppose that there is another monoid homomorphism $g : A^* \to N$ such that $g \circ i = f$. For every $w = a_1 \ldots a_n \in A^*$ we have

$$
g(w) = g(a_1 \cdots a_n) = g(a_1) \cdot \cdots \cdot g(a_n) = f(a_1) \cdot \cdots \cdot f(a_n) = \bar{f}(a_1 \cdots a_n),
$$

hence $\bar{f} = g$. $\square$

Corollary 1.5. Universal mapping property of the free monoid determines it uniquely up to an isomorphism.

Proof. Suppose that M and N are free monoids on a set A. Consider the inclusion homomorphisms $i : A \to M$ and $j : A \to N$. Since M is a free monoid, by universal mapping property there is a monoid homomorphism $\alpha : M \to N$ such that $\alpha \circ i = j$. Since N is a free monoid, by universal mapping property there is a monoid homomorphism $\beta : N \to M$ such that $\beta \circ j = i$. It follows that $(\beta \circ \alpha) \circ i = i$. But we also have $1_M \circ i = i$. Then by the uniqueness from universal mapping property we must have $\beta \circ \alpha = 1_M$.

$$
\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "j"'] & M \arrow[dl, dashed, "\alpha"] \\
N &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "j"] \arrow[d, "i"'] & N \arrow[dl, dashed, "\beta"] \\
M &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "i"'] & M \arrow[dl, dashed, "\beta\circ\alpha=1_M"] \\
M &
\end{tikzcd}
$$

Similarly, we have $\alpha \circ \beta = 1_N$. Hence $\alpha : M \to N$ is a monoid isomorphism. $\square$

Next let us see how can we generalize the above results to categories.

To each category $\mathcal{C}$ we may associate a graph $G = (V, E)$, where the class V of vertices consists of the objects of $\mathcal{C}$, while the class E of edges consists of the arrows of $\mathcal{C}$. Then we have two functions $s : E \to V$ (source) defined by $s(e) = v_1$ for every arrow $e : v_1 \to v_2$, and $t : E \to V$ (target) defined by $t(e) = v_2$ for every arrow $e : v_1 \to v_2$.

We define the free category on the graph G, denoted by $\mathcal{C}(G)$, as follows:

- objects: the vertices of G
- arrows: the paths in G
- composition: the concatenation of paths in G
- identity arrow: for every $v \in V$ the identity arrow 1_v is the loop on v.

We may define a functor $U : \mathbf{Cat} \to \mathbf{Graph}$, called the **forgetful functor**, which associates to a category $\mathcal{C}$ its underlying graph having as edges the arrows of $\mathcal{C}$, and as vertices the objects of $\mathcal{C}$, and to a functor between categories its underlying graph homomorphism (that is, a functor without the conditions on composition and identity). One may prove the following result.

Theorem 1.6 (Universal Mapping Property of the Free Category on a Graph)**.** With the above notation, there is a graph homomorphism $i : G \to U(\mathcal{C}(G))$ with the property that for every category $\mathcal{D}$ and for every graph homomorphism $f : G \to U(\mathcal{D})$, there is a unique functor $\bar{f} : \mathcal{C}(G) \to \mathcal{D}$ such that $U(\bar{f}) \circ i = f$, that is, the following diagram is commutative:

$$\begin{tikzcd} G \arrow[r, "i"] \arrow[d, "f"'] & U(\mathcal{C}(G)) \arrow[dl, dashed, "U(\bar{f})"] \\ U(\mathcal{D}) & \end{tikzcd}$$

Corollary 1.7. Universal mapping property of the free category on a graph determines it uniquely up to a graph isomorphism.

1.7 Large, Small and Locally Small Categories

Definition 1.4. A category is called:

(1) **small** if both classes of objects and arrows are sets.

(2) **large** if it is not small.

(3) **locally small** if for every objects $C, D \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(C, D)$ is a set.

Example 1.1.

(1) The category of finite sets and functions is equivalent to a small category.

(2) **Set**, **Pos**, **Grp** and **Top** are locally small, but not small.

(3) **Cat** is large, but not locally small. Indeed, if $\mathcal{C}$ is a locally small category which is not small, and $\mathbf{1}$ is the category with one object and one arrow, then functors $\mathbf{1} \to \mathcal{C}$ are simply objects of $\mathcal{C}$, so $\mathrm{Hom}_{\mathbf{Cat}}(1, \mathcal{C})$ is not a set.

Chapter 2

Abstract Structure

2.1 Epimorphisms and Monomorphisms

Definition 2.1. Let $\mathcal{C}$ be a category. An arrow $f : A \to B$ in $\mathcal{C}$ is called a:

(1) **monomorphism** (or briefly *mono*) if for every arrows $g, h : C \to A$ such that $f \circ g = f \circ h$, we have $g = h$.

(2) **epimorphism** (or briefly *epi*) if for every arrows $g, h : B \to C$ such that $g \circ f = h \circ f$, we have $g = h$.

(3) **bimorphism** if it is both a monomorphism and an epimorphism.

Lemma 2.1. Every isomorphism is a bimorphism.

Proof. Let $f : A \to B$ be an isomorphism in a category $\mathcal{C}$. Let $g, h : C \to A$ be arrows in $\mathcal{C}$ such that $f \circ g = f \circ h$. Compose on the left by f^{-1} in order to get $g = h$. Hence f is a monomorphism. Similarly, one shows that f is an epimorphism. Thus, f is a bimorphism. $\square$

One may also prove the following property.

Proposition 2.2. Let $f : A \to B$ and $g : B \to C$ be arrows in a category $\mathcal{C}$.

(1) If f, g are monomorphisms (epimorphisms), then so is $g \circ f$.

(2) If $g \circ f$ is monomorphism, then so is f.

(3) If $g \circ f$ is epimorphism, then so is g.

Example 2.1.

(1) In **Set** monomorphisms coincide with injective functions, epimorphisms coincide with surjective functions, and bimorphisms coincide with isomorphisms and with bijective functions.

(2) In many usual concrete categories, monomorphisms coincide with injective arrows. In not so many usual categories epimorphisms coincide with surjective arrows.

For instance, in **Mon** monomorphisms coincide with injective monoid homomorphisms, but epimorphisms do not coincide with surjective monoid homomorphisms. Indeed, the inclusion map $i : \mathbb{N} \to \mathbb{Z}$, defined by $i(x) = x$, is an epimorphism in **Mon**, but it is not surjective. This is also an example of a bimorphism, which is not an isomorphism.

(3) There is a large class of categories, called **abelian categories** (e.g., the categories **Ab** and $\mathrm{Vect}(K)$), in which bimorphisms coincide with isomorphisms.

Definition 2.2. Let $\mathcal{C}$ be a category. An arrow $f : A \to B$ in $\mathcal{C}$ is called a:

(1) **section** (or **split monomorphism**) if it has a left inverse arrow, that is, there is an arrow $g : B \to A$ such that $g \circ f = 1_A$.

(2) **retraction** (or **split epimorphism**) if it has a right inverse arrow, that is, there is an arrow $g : B \to A$ such that $f \circ g = 1_B$.

Remark. An arrow is an isomorphism if and only if it is a section and a retraction.

Lemma 2.3.

(1) Every section is a monomorphism.

(2) Every retraction is an epimorphism.

Proof.

(1) Let $f : A \to B$ be a section in a category $\mathcal{C}$. Hence there is an arrow $r : B \to A$ such that $r \circ f = 1_A$. Let $g, h : B \to C$ be arrows in $\mathcal{C}$ such that $f \circ g = f \circ h$. Then we have

$$r \circ f \circ g = r \circ f \circ h,$$

which implies that $g = h$. Hence f is a monomorphism.

(2) Similarly, one shows that every retraction is an epimorphism.

□

One may show the following property.

Proposition 2.4. The following are equivalent for an arrow $f : A \to B$ in any category $\mathcal{C}$:

(1) f is an isomorphism.

(2) f is both a monomorphism and a retraction.

(3) f is both a section and an epimorphism.

Example 2.2.

(1) In **Set** every monomorphism (i.e., injective function) is a section, except those of the form $\varnothing \to A$ with $A \neq \varnothing$. In **Set** the condition that every epimorphism is a retraction is equivalent to the axiom of choice.

(2) In **Ab** the inclusion homomorphism $i : 2\mathbb{Z} \to \mathbb{Z}$, defined by $i(2x) = 2x$, is clearly a monomorphism. Suppose that it is a section. Then there is a group homomorphism $g : \mathbb{Z} \to 2\mathbb{Z}$ such that $g \circ i = 1_{2\mathbb{Z}}$. We have $2g(1) = g(2) = g(i(2)) = 2$, hence $1 = g(1) \in 2\mathbb{Z}$, a contradiction. Therefore, there are monomorphisms, which are not sections.

In **Ab** the homomorphism $f : \mathbb{Z}_4 \to \mathbb{Z}_2$ defined by $f(\bar{x}) = \hat{x}$ is clearly an epimorphism. Suppose that it is a retraction. Then there is a group homomomorphism $g : \mathbb{Z}_2 \to \mathbb{Z}_4$ such that $f \circ g = 1_{\mathbb{Z}_2}$. The order of $g(\hat{1})$ divides the order of $\hat{1}$, which is 2. Hence $g(\hat{1}) \in \{\bar{0}, \bar{2}\}$. But then $f(g(\hat{1})) = \hat{0} \neq \hat{1} = 1_{\mathbb{Z}_2}(\hat{1})$, a contradiction. Therefore, there are epimorphisms which are not retractions.

Proposition 2.5. Let $F : \mathcal{C} \to \mathcal{D}$ be a (covariant) functor. Then:

(1) F preserves sections in the sense that if f is a section, then $F(f)$ is also a section.

(2) F preserves retractions in the sense that if f is a retraction, then $F(f)$ is also a retraction.

(3) F preserves isomorphisms in the sense that if f is an isomorphism, then $F(f)$ is also an isomorphism.

Proof. Let $f : A \to B$ be a section in $\mathcal{C}$. Then there is an arrow $g : B \to A$ such that $g \circ f = 1_A$. This implies that

$$F(g) \circ F(f) = F(g \circ f) = F(1_A) = 1_{F(A)}$$

hence $F(f)$ is a section in $\mathcal{D}$. Similarly, F preserves retractions. The fact that F preserves isomorphisms is a consequence of the first two properties. □

2.2 Initial and Terminal Objects

Definition 2.3. Let $\mathcal{C}$ be a category. An object C' of $\mathcal{C}$ is called:

(1) **initial** if for every object C of $\mathcal{C}$, there is a unique arrow $C' \to C$.

(2) **terminal** if for every object C of $\mathcal{C}$, there is a unique arrow $C \to C'$.

Sometimes an initial object is denoted by 0, while a terminal object is denoted by 1.

Proposition 2.6. Initial and terminal objects are unique up to an isomorphism.

Proof. Assume that C', C'' are initial objects of a category $\mathcal{C}$. Then there is a unique arrow $f : C' \to C''$ and a unique arrow $g : C'' \to C'$. Note that we have the arrows $g \circ f : C' \to C'$ and $1_{C'} : C' \to C'$. Since C' is an initial object, we must have $g \circ f = 1_{C'}$. Also, note that we have the arrows $f \circ g : C'' \to C''$ and $1_{C''} : C'' \to C''$. Since C'' is an initial object, we must have $f \circ g = 1_{C''}$. Hence $f : C' \to C''$ is an isomorphism. Hence initial objects are unique up to an isomorphism.

Similarly, one shows that terminal objects are unique up to an isomorphism. □

Example 2.3.

(1) In **Set** the initial object is $\varnothing$, while the terminal object is any single-element set.

(2) In **Grp** the initial object is the trivial group, while the terminal object is again the trivial group.

(3) In **Ring** the initial object is $\mathbb{Z}$, while the terminal object is the trivial ring. Note that there is a unique unitary ring homomorphism $\mathbb{Z} \to R$ for every ring R with identity $1'$, which is defined by

$$f(n) = n \cdot 1', \quad \forall x \in \mathbb{Z}$$

We first show that if f does exist, then it is unique. So, suppose that $f : \mathbb{Z} \to R$ is a unitary ring homomorphism. Then $f(0) = 0' = 0 \cdot 1'$, where $0'$ is the zero element of R. For every $k \in \mathbb{N}^*$, we have:

$$f(k) = f(\underbrace{1 + \cdots + 1}_{k \text{ times}}) = \underbrace{f(1) + \cdots + f(1)}_{k \text{ times}} = \underbrace{1' + \cdots + 1'}_{k \text{ times}} = k \cdot 1'$$
$$f(-k) = -f(k) = -(k \cdot 1') = (-k) \cdot 1'$$

Hence $f(n) = n \cdot 1'$ for every $n \in \mathbb{Z}$.

Now we show that the function f is a unitary ring homomorphism. For every $m, n \in \mathbb{Z}$, we have:

$$f(m + n) = (m + n) \cdot 1' = m \cdot 1' + n \cdot 1' = f(m) + f(n),$$
$$f(m \cdot n) = (m \cdot n) \cdot 1' = (m \cdot 1') \cdot (n \cdot 1') = f(m) \cdot f(n)$$

and $f(1) = 1 \cdot 1' = 1'$. Hence f is a unitary ring homomorphism.

(4) View the poset $(\mathbb{Z}, \leqslant)$ as a poset category. This category has neither an initial object, nor a terminal object.

2.3 Products

Definition 2.4. Let $\mathcal{C}$ be a category and A, B objects of $\mathcal{C}$. A **product diagram** for A and B consists of an object P and arrows, called **canonical projections**

$$A \xleftarrow{p_1} P \xrightarrow{p_2} B$$

satisfying the following universal mapping property: given any diagram of the form

$$A \xleftarrow{f_1} X \xrightarrow{f_2} B$$

there is a unique arrow $u : X \to P$ such that

$$p_1 \circ u = f_1 \quad \text{and} \quad p_2 \circ u = f_2$$

that is, the following diagram is commutative:

$$\begin{tikzcd} & X \arrow[dl, "f_1"'] \arrow[d, dashed, "u"] \arrow[dr, "f_2"] & \\ A & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & B \end{tikzcd}$$

We denote the product of A and B by (P, p_1, p_2). Sometimes P is also denoted by $A \prod B$ or $A \times B$.

Remark. Sometimes (especially in the so-called additive categories), the canonical projections $p_1 : A \times B \to A$ and $p_2 : A \times B \to B$ are also denoted by $\begin{bmatrix} 1 & 0 \end{bmatrix} : A \times B \to A$ and $\begin{bmatrix} 0 & 1 \end{bmatrix} : A \times B \to B$ respectively. The unique arrow $u : X \to A \times B$ such that $p_2 \circ u = f_1$ and $p_2 \circ u = f_2$ is also denoted by

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} : X \to A \times B$$

Then equalities involving compositions of arrows such as $p_1 \circ u = f_1$ and $p_2 \circ u = f_2$ may be rewritten in terms of matrix multiplications as $\begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = f_1$ and $\begin{bmatrix} 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = f_2$.

Remark. Note that the canonical projections p_1 and p_2 need not be epimorphisms. For instance, consider the category described by the following graph:

$$\begin{tikzcd} A & P \arrow[l, "p_1"'] \arrow[r, "p_2"] & B \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & C \end{tikzcd}$$

such that $g \circ p_2 = h \circ p_2$. Then (P, p_1, p_2) is a product of A and B, but p_2 is not an epimorphism, because we have $g \circ p_2 = h \circ p_2$ and $g \neq h$.

As usual, universal mapping property insures the following uniqueness result.

Proposition 2.7. The product is unique up to an isomorphism.

Proof. Suppose that (P, p_1, p_2) and (Q, q_1, q_2) are products of objects A and B of a category $\mathcal{C}$. Since (Q, q_1, q_2) is a product, there is a unique arrow $i : P \to Q$ such that $q_1 \circ i = p_1$ and $q_2 \circ i = p_2$. Since (P, p_1, p_2) is a product, there is a unique arrow $j : Q \to P$ such that $p_1 \circ j = q_1$ and $p_2 \circ j = q_2$. Hence we have the following commutative diagram:

$$\begin{tikzcd} & P \arrow[dl, "p_1"'] \arrow[d, dashed, "i"] \arrow[dr, "p_2"] & \\ A & Q \arrow[l, "q_1"'] \arrow[r, "q_2"] \arrow[d, dashed, "j"] & B \\ & P \arrow[ul, "p_1"] \arrow[ur, "p_2"'] & \end{tikzcd}$$

It follows that $p_1 \circ (j \circ i) = p_1$ and $p_2 \circ (j \circ i) = p_2$. But we also have $p_1 \circ 1_P = p_1$ and $p_2 \circ 1_P = p_2$, and by the uniqueness condition of universal mapping property we have $j \circ i = 1_P$. Similarly, one shows that $i \circ j = 1_Q$. Hence $i : P \to Q$ is an isomorphism. $\square$

More generally, one may define a product of an arbitrary family of objects of a category, which is again unique up to an isomorphism.

Definition 2.5. A **product** of a family $(A_i)_{i \in I}$ of objects of a category $\mathcal{C}$ consists of an object P, also denoted by $\prod_{i \in I} A_i$, and a family $(p_i)_{i \in I}$ of arrows, where $p_i : P \to A_i$ for every $i \in I$, satisfying the following universal mapping property: given any object X of $\mathcal{C}$ and any family $(f_i)_{i \in I}$ of arrows, where $f_i : X \to A_i$ for every $i \in I$, there is a unique arrow $u : X \to P$ such that $p_i \circ u = f_i$ for every $i \in I$.

2.4 Examples of Products

2.4.1 The Category Set

The product of two sets A and B is

$$(A \times B, p_1, p_2)$$

where $p_1 : A \times B \to A$ is the function defined by $p_1(a, b) = a$, and $p_2 : A \times B \to B$ is the function defined by $p_2(a, b) = b$.

Let X be a set and let $f_1 : X \to A$ and $f_2 : X \to B$ be functions. We look for a unique function $u : X \to A \times B$ such that the following diagram is commutative:

$$\begin{tikzcd} & X \arrow[dl, "f_1"'] \arrow[d, dashed, "u"] \arrow[dr, "f_2"] & \\ A & A \times B \arrow[l, "p_1"] \arrow[r, "p_2"'] & B \end{tikzcd}$$

that is, $p_1 \circ u = f_1$ and $p_2 \circ u = f_2$. These equalities are equivalent to $p_1(u(x)) = f_1(x)$ and $p_2(u(x)) = f_2(x)$ for every $x \in X$. This means that

$$u(x) = (f_1(x), f_2(x)), \quad \forall x \in X$$

Note that u is uniquely determined by this definition.

The construction of a product may be easily generalized to an arbitrary family of sets.

2.4.2 The Category Grp

The product of two groups $(G_1, \cdot)$ and $(G_2, \cdot)$ is

$$((G_1 \times G_2, \cdot), p_1, p_2)$$

where $p_1 : G_1 \times G_2 \to G_1$ is the function defined by $p_1(g_1, g_2) = g_1$, and $p_2 : G_1 \times G_2 \to G_2$ is the function defined by $p_2(g_1, g_2) = g_2$.

Note that $G_1 \times G_2$ is a group with respect to the operation defined by

$$(x_1, x_2) \cdot (x_1', x_2') = (x_1 \cdot x_1', x_2 \cdot x_2'), \quad \forall (x_1, x_2), (x_1', x_2') \in G_1 \times G_2$$

Let $(X, \cdot)$ be a group and let $f_1 : X \to G_1$ and $f_2 : X \to G_2$ be group homomorphisms. We look for a unique group homomorphism $u : X \to G_1 \times G_2$ such that the following diagram is commutative:

$$\begin{tikzcd} & X \arrow[dl, "f_1"'] \arrow[d, dashed, "u"] \arrow[dr, "f_2"] & \\ G_1 & G_1 \times G_2 \arrow[l, "p_1"] \arrow[r, "p_2"'] & G_2 \end{tikzcd}$$

that is, $p_1 \circ u = f_1$ and $p_2 \circ u = f_2$. These equalities are equivalent to $p_1(u(x)) = f_1(x)$ and $p_2(u(x)) = f_2(x)$ for every $x \in X$. This means that

$$u(x) = (f_1(x), f_2(x)), \quad \forall x \in X$$

Note that u is uniquely determined by this definition.

We still need to prove that u is a group homomorphism. For every $x_1, x_2 \in X$ we have

$$\begin{aligned} u(x_1 \cdot x_2) &= (f_1(x_1 \cdot x_2), f_2(x_1 \cdot x_2)) = (f_1(x_1) \cdot f_1(x_2), f_2(x_1) \cdot f_2(x_2)) \\ &= (f_1(x_1), f_2(x_1)) \cdot (f_1(x_2), f_2(x_2)) = u(x_1) \cdot u(x_2) \end{aligned}$$

hence u is a group homomorphism.

The construction of a product may be easily generalized to an arbitrary family of groups.

2.4.3 Poset Categories

Let $(L, \leqslant)$ be a lattice. Hence every two elements of L have an infimum (i.e., greatest lower bound). Since $(L, \leqslant)$ is a poset, we may view it as a poset category. Recall that its objects are the elements of L, while an arrow $x \to y$ does exists if and only if $x \leqslant y$, where $x, y \in L$.

The product of two elements $x, y \in L$ is

$$(\inf(x, y), p_1, p_2)$$

where $p_1 : \inf(x, y) \to x$ and $p_2 : \inf(x, y) \to y$ are the unique arrows having the given domains and codomains.

Let $z \in L$ and let $z \to x$ and $z \to y$ be arrows. This means that $z \leq x$ and $z \leq y$. We look for a unique arrow $u : z \to \inf(x, y)$ such that the following diagram is commutative:

$$\begin{tikzcd} & z \arrow[dl] \arrow[d, dashed, "u"] \arrow[dr] & \\ x & \inf(x, y) \arrow[l] \arrow[r] & y \end{tikzcd}$$

This means that $z \leqslant \inf(x, y)$. But this is true, because z is a lower bound of x and y, while $\inf(x, y)$ is the greatest lower bound of x and y.

Note that if a poset $(L, \leqslant)$ is not a lattice, two elements of L might not have a product.

The construction of a product may be easily generalized to an arbitrary family of elements, when $(L, \leqslant)$ is a complete lattice, that is, every family of elements of L has an infimum and a supremum.

2.5 Categories with Products

Definition 2.6. A category $\mathcal{C}$ is said to **have (binary) products** if any family of (two) objects of $\mathcal{C}$ has a product.

Example 2.4. We have seen that **Set** and **Grp** have (binary) products, while poset categories may not have products.

Let $\mathcal{C}$ be a category with binary products, and let $f : A \to B$ and $f' : A' \to B'$ be arrows in $\mathcal{C}$. We want to define a product $f \times f' : A \times A' \to B \times B'$ of f and f'.

Consider the products $(A \times A', p_1, p_2)$ and $(B \times B', q_1, q_2)$. Let $f_1 = f \circ p_1$ and $f_2 = f' \circ p_2$. By universal mapping property of the product $(B \times B', q_1, q_2)$ there is a unique arrow $u : A \times A' \to B \times B'$ such that the following diagram is commutative:

$$\begin{tikzcd} A \arrow[d, "f"'] & A \times A' \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, dashed, "u"] & A' \arrow[d, "f'"] \\ B & B \times B' \arrow[l, "q_1"] \arrow[r, "q_2"'] & B' \end{tikzcd}$$

that is, $q_1 \circ u = f_1 = f \circ p_1$ and $q_2 \circ u = f_2 = f' \circ p_2$. We define $f \times f' = u : A \times A' \to B \times B'$.

One may prove the following result.

Proposition 2.8. Let $\mathcal{C}$ be a category with binary products. Then we have a covariant functor $R : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ defined by

$$R\,(C, C') = C \times C'$$

for every object (C, C') of $\mathcal{C} \times \mathcal{C}$, and

$$R\,(f, f') = f \times f' : A \times A' \to B \times B'$$

for every arrow (f, f') from $\mathcal{C} \times \mathcal{C}$ with $f : A \to B$ and $f' : A' \to B'$.

For a category $\mathcal{C}$ with products, one may generalize this construction to any finite family of arrows, and define a corresponding functor.

One may show the following associativity property by using universal mapping property of the product.

Proposition 2.9. In any category $\mathcal{C}$ with binary products, we have

$$A \times (B \times C) \cong (A \times B) \times C,$$

where A, B, C are objects of $\mathcal{C}$.

2.6 Hom-sets

In this section, assume that all categories are locally small, that is, $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is a set for every $A, B \in \mathcal{C}$.

Let A be an object and $f : B \to B'$ an arrow in a category $\mathcal{C}$. We define

$$\begin{aligned} &f_* = \mathrm{Hom}_{\mathcal{C}}(A, f) : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}\,(A, B') \\ &f_*(g) = f \circ g, \quad \forall g \in \mathrm{Hom}_{\mathcal{C}}(A, B) \end{aligned}$$

One may show the following property.

Proposition 2.10. Let $\mathcal{C}$ be a category and let A be an object of $\mathcal{C}$. Then we have a covariant functor $\mathrm{Hom}_{\mathcal{C}}(A,-):\mathcal{C}\to\mathbf{Set}$, called the covariant representable functor, defined by

$$B \mapsto \mathrm{Hom}_{\mathcal{C}}(A,B)$$

on every object B of $\mathcal{C}$ and

$$f \mapsto f_* = \mathrm{Hom}_{\mathcal{C}}(A,f) : \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{C}}\left(A,B'\right)$$

for every arrow $f : B \to B'$ in $\mathcal{C}$.

One may show the following property.

Proposition 2.11. Let $\mathcal{C}$ be a category with binary products. Then for every object A of $\mathcal{C}$, the covariant functor $\mathrm{Hom}_{\mathcal{C}}(A,-):\mathcal{C}\to\mathbf{Set}$ preserves binary products, that is, for every $C, D \in \mathcal{C}$, there is a bijection (i.e., isomorphism in **Set**):

$$\mathrm{Hom}_{\mathcal{C}}(A, C\times D) \cong \mathrm{Hom}_{\mathcal{C}}(A,C)\times \mathrm{Hom}_{\mathcal{C}}(A,D).$$

Definition 2.7. A contravariant functor between two categories $\mathcal{C}$ and $\mathcal{D}$ is a mapping of objects of $\mathcal{C}$ to objects of $\mathcal{D}$ and of arrows of $\mathcal{C}$ to arrows of $\mathcal{D}$, denoted by

$$F : \mathcal{C} \to \mathcal{D}$$

satisfying the axioms:

(i) For every $f : A \to B$ in $\mathcal{C}$, we have $F(f) : F(B) \to F(A)$ in $\mathcal{D}$.

(ii) For every object A of $\mathcal{C}$, we have $F\left(1_A\right) = 1_{F(A)}$.

(iii) For every composable pair of arrows $f : A \to B$ and $g : B \to C$ in $\mathcal{C}$, we have

$$F(g\circ f) = F(f)\circ F(g)$$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} F(A) & F(B) \arrow[l, "F(f)"'] \\ & F(C) \arrow[u, "F(g)"'] \arrow[ul, "F(g\circ f)"] \end{tikzcd}$$

Let A be an object and $f : B \to B'$ an arrow in a category $\mathcal{C}$. We define

$$f^* = \mathrm{Hom}_{\mathcal{C}}(f,A) : \mathrm{Hom}_{\mathcal{C}}\left(B',A\right) \to \mathrm{Hom}_{\mathcal{C}}(B,A),$$
$$f^*(g) = g\circ f, \quad \forall g \in \mathrm{Hom}_{\mathcal{C}}\left(B',A\right).$$

Proposition 2.12. Let $\mathcal{C}$ be a category and let A be an object of $\mathcal{C}$. Then we have a contravariant functor $\mathrm{Hom}_{\mathcal{C}}(-,A):\mathcal{C}\to\mathbf{Set}$, called the **contravariant representable functor**, defined by

$$B \mapsto \mathrm{Hom}_{\mathcal{C}}(B,A)$$

on every object B of $\mathcal{C}$ and

$$f \mapsto f^* = \mathrm{Hom}_{\mathcal{C}}(f,A) : \mathrm{Hom}_{\mathcal{C}}\left(B',A\right) \to \mathrm{Hom}_{\mathcal{C}}(B,A)$$

on every arrow $f : B \to B'$ in $\mathcal{C}$.

Proof.

(i) For every arrow $f : B \to B'$ in $\mathcal{C}$, we have the following function (arrow in **Set**):

$$f^* = \mathrm{Hom}_{\mathcal{C}}(f, A) : \mathrm{Hom}_{\mathcal{C}}(B', A) \to \mathrm{Hom}_{\mathcal{C}}(B, A)$$

(ii) For every object B of $\mathcal{C}$, we have

$$\mathrm{Hom}_{\mathcal{C}}(1_B, A) = 1_B^* = 1^{\mathcal{C}}_{\mathrm{Hom}}(B, A)$$

(iii) Let $f : B \to B'$ and $g : B' \to B''$ be arrows in $\mathcal{C}$. Then we have the following functions (arrows in **Set**)

$$(g \circ f)^*, f^* \circ g^* : \mathrm{Hom}_{\mathcal{C}}(B'', A) \to \mathrm{Hom}_{\mathcal{C}}(B, A)$$

For every function $h \in \mathrm{Hom}_{\mathcal{C}}(B'', A)$, we have

$$(g \circ f)^*(h) = h \circ (g \circ f) = (h \circ g) \circ f = g^*(h) \circ f = f^*(g^*(h)) = (f^* \circ g^*)(h)$$

Hence we have $(g \circ f)^* = f^* \circ g^*$. It follows that

$$\mathrm{Hom}_{\mathcal{C}}(g \circ f, A) = (g \circ f)^* = f^* \circ g^* = \mathrm{Hom}_{\mathcal{C}}(f, A) \circ \mathrm{Hom}_{\mathcal{C}}(g, A)$$

Hence $\mathrm{Hom}_{\mathcal{C}}(-, A) : \mathcal{C} \to$ **Set** is a contravariant functor. □

Chapter 3

Duality

3.1 The Duality Principle

In the formal definition of a category there are objects $A, B, C, \ldots$, arrows $f, g, h, \ldots$ and four operations given by $\mathrm{dom}(f), \mathrm{cod}(f), 1_A, g \circ f$ which satisfy the following seven axioms:

- $\mathrm{dom}\,(1_A) = A$.
- $\mathrm{cod}\,(1_A) = A$.
- $f \circ 1_{\mathrm{dom}(f)} = f$.
- $1_{\mathrm{cod}(f)} \circ f = f$.
- $\mathrm{dom}(g \circ f) = \mathrm{dom}(f)$.
- $\mathrm{cod}(g \circ f) = \mathrm{cod}(g)$.
- $h \circ (g \circ f) = (h \circ g) \circ f$.

Of course, the operation "$g \circ f$" is only defined when $\mathrm{dom}(g) = \mathrm{cod}(f)$.

Given any sentence Σ in the elementary language of category theory, we can form the "dual statement" Σ^* by making the following replacements: $f \circ g$ for $g \circ f$, cod for dom, dom for cod. It is easy to see that then Σ^* will also be a well-formed sentence. Next, suppose we have shown a sentence Σ to entail one Δ, that is,

$$\Sigma \Longrightarrow \Delta,$$

without using any of the category axioms, then clearly

$$\Sigma^* \Longrightarrow \Delta^*$$

since the substituted terms are treated as mere undefined constants. But now observe that the axioms for category theory (CT) are themselves "self-dual" in the sense that we have $CT^* = CT$.

Therefore we have the following formal duality principle.

Proposition 3.1 (Formal duality)**.** For any sentence Σ in the language of category theory (CT) if Σ follows from the axioms of categories, then its dual Σ^* also follows, i.e.,

$$(CT \Longrightarrow \Sigma) \text{ implies } (CT \Longrightarrow \Sigma^*)$$

Now assume that Σ holds for any category $\mathcal{C}$. Then Σ holds for any opposite category $\mathcal{C}^{\mathrm{op}}$. Hence Σ^* holds in $\mathcal{C} = (\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ for any category $\mathcal{C}$.

Therefore we have the following conceptual form of the duality principle.

Proposition 3.2 (Conceptual duality)**.** For any statement Σ about categories, if Σ holds for all categories, then Σ^* holds for all categories.

3.2 Coproducts

Definition 3.1. Let $\mathcal{C}$ be a category and let A and B be objects of $\mathcal{C}$. A coproduct of A and B in $\mathcal{C}$ is just the product of A and B in the opposite category $\mathcal{C}^{\mathrm{op}}$. This means an object Q and arrows $q_1 : A \to Q$ and $q_2 : B \to Q$, called canonical injections, satisfying the following universal mapping property: given any diagram of the form

$$A \xrightarrow{f_1} Z \xleftarrow{f_2} B$$

there is a unique arrow $u : Q \to Z$ such that

$$u \circ q_1 = f_1 \quad \text{and} \quad u \circ q_2 = f_2$$

that is, the following diagram is commutative:

$$\begin{tikzcd} & Z & \\ A \arrow[ur, "f_1"] \arrow[r, "q_1"'] & Q \arrow[u, dashed, "u"'] & B \arrow[ul, "f_2"'] \arrow[l, "q_2"] \end{tikzcd}$$

We denote the coproduct of A and B by (Q, q_1, q_2). Sometimes Q is also denoted by $A \coprod B$ or $A \oplus B$.

Remark. Sometimes (especially in the so-called additive categories), similarly to the case of products, the canonical injections $q_1 : A \to A \oplus B$ and $q_2 : B \to A \oplus B$ are also denoted by $\begin{bmatrix} 1 \\ 0 \end{bmatrix} : A \to A \oplus B$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix} : B \to A \oplus B$ respectively. The unique arrow $u : A \oplus B \to Z$ such that $u \circ q_1 = f_1$ and $u \circ q_2 = f_2$ is also denoted by

$$\begin{bmatrix} f_1 & f_2 \end{bmatrix} : A \oplus B \to Z$$

Then equalities involving compositions of arrows such as $u \circ q_1 = f_1$ and $u \circ q_2 = f_2$ may be rewritten in terms of matrix multiplications as $\begin{bmatrix} f_1 & f_2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = f_1$ and $\begin{bmatrix} f_1 & f_2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = f_2$.

Remark. Note that the canonical injections q_1 and q_2 need not be monomorphisms. Just consider the example with the canonical projections from products in the opposite category.

As usual, universal mapping property insures the following uniqueness result. It follows by the duality principle.

Proposition 3.3. The coproduct is unique up to an isomorphism.

More generally, one may define a coproduct of an arbitrary family of objects of a category, which is again unique up to an isomorphism.

Definition 3.2. A coproduct of a family $(A_i)_{i\in I}$ of objects of a category $\mathcal{C}$ consists of an object Q, also denoted by $\coprod\limits_{i\in I} A_i$ or $\bigoplus\limits_{i\in I} A_i$, and a family $(q_i)_{i\in I}$ of arrows, where $q_i : A_i \to Q$ for every $i \in I$, satisfying the following universal mapping property: given any object Z of $\mathcal{C}$ and any family $(f_i)_{i\in I}$ of arrows, where $f_i : A_i \to Z$ for every $i \in I$, there is a unique arrow $u : Q \to Z$ such that $u \circ q_i = f_i$ for every $i \in I$.

Next we present some examples of coproducts in certain categories.

3.2.1 The Category Set

The coproduct of two sets A and B is $(A \sqcup B, q_1, q_2)$, where

$$A \sqcup B = \{(a,1) \mid a \in A\} \cup \{(b,2) \mid b \in B\}$$

is the disjoint union of A and $B, q_1 : A \to A \sqcup B$ is the function defined by $q_1(a) = (a,1)$, and $q_2 : B \to A \sqcup B$ is the function defined by $q_2(b) = (b,2)$.

Let Z be a set and let $f_1 : A \to Z$ and $f_2 : B \to Z$ be functions. We look for a unique function $u : A \sqcup B \to Z$ such that the following diagram is commutative:

$$\begin{tikzcd} & Z & \\ A \arrow[ur, "f_1"] \arrow[r, "q_1"'] & A \sqcup B \arrow[u, dashed, "u"'] & B \arrow[ul, "f_2"'] \arrow[l, "q_2"] \end{tikzcd}$$

that is, $u \circ q_1 = f_1$ and $u \circ q_2 = f_2$. These equalities are equivalent to $u\,(q_1(a)) = f_1(a)$ and $u\,(q_2(b)) = f_2(b)$ for every $a \in A$ and $b \in B$, and furthermore, $u(a,1) = f_1(a)$ and $u(b,2) = f_2(b)$ for every $a \in A$ and $b \in B$. Note that u is uniquely determined by this definition.

The construction of a coproduct may be easily generalized to an arbitrary family of sets.

3.2.2 The Category Ab

The coproduct of two abelian groups $(A,+)$ and $(B,+)$ is $((A \times B,+), q_1, q_2)$, where $q_1 : A \to A \times B$ is the group homomorphism defined by $q_1(a) = (a,0)$, and $q_2 : B \to A \times B$ is the group homomorphism defined by $q_2(b) = (0,b)$.

Note that $A \times B$ is a group with respect to the operation defined by

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)\,, \quad \forall\,(a_1, b_1)\,, (a_2, b_2) \in A \times B.$$

Let $(Z,+)$ be an abelian group and let $f_1 : A \to Z$ and $f_2 : B \to Z$ be group homomorphisms. We look for a unique group homomorphism $u : A \times B \to Z$ such that the following diagram is commutative:

$$\begin{tikzcd} & Z & \\ A \arrow[ur, "f_1"] \arrow[r, "q_1"'] & A \times B \arrow[u, dashed, "u"] & B \arrow[ul, "f_2"'] \arrow[l, "q_2"] \end{tikzcd}$$

that is, $u \circ q_1 = f_1$ and $u \circ q_2 = f_2$. These equalities are equivalent to $u\left(q_1(a)\right) = f_1(a)$ and $u\left(q_2(b)\right) = f_2(b)$ for every $a \in A$ and $b \in B$, and furthermore, $u(a, 0) = f_1(a)$ and $u(0, b) = f_2(b)$ for every $a \in A$ and $b \in B$. Note that a group homomorphism u is uniquely determined by this definition, because we have

$$u(a, b) = u((a, 0) + (0, b)) = u(a, 0) + u(0, b) = f_1(a) + f_2(b)$$

for every $a \in A$ and $b \in B$. One checks that the map u defined as above is really a group homomorphism.

The construction of a coproduct may be generalized to an arbitrary family of abelian groups, but in a slightly different manner. For a family $(A_i)_{i \in I}$ of abelian groups, we denote

$$\bigoplus_{i \in I} A_i = \left\{ (a_i)_{i \in I} \in \prod_{i \in I} A_i \mid (a_i)_{i \in I} \text{ has a finite number of non-zero elements} \right\}.$$

Note that if I is a finite set, then we have $\bigoplus\limits_{i \in I} A_i \cong \prod_{i \in I} A_i$.

The coproduct of the family $(A_i)_{i \in I}$ of abelian groups is $\left(\bigoplus\limits_{i \in I} A_i, (q_i)_{i \in I}\right)$, where for every $j \in I, q_j : A_j \to \bigoplus\limits_{i \in I} A_i$ is the group homomorphism defined by $q_j(a) = (a_i)_{i \in I}$ with the properties that $a_j = a$ and $a_i = 0$ for every $i \in I$ with $i \neq j$.

3.2.3 Poset Categories

Let $(L, \leqslant)$ be a lattice. Hence every two elements of L have a supremum (i.e., smallest upper bound). Since $(L, \leqslant)$ is a poset, we may view it as a poset category. Recall that its objects are the elements of L, while an arrow $x \to y$ does exists if and only if $x \leqslant y$, where $x, y \in L$.

The coproduct of two elements $x, y \in L$ is $(\sup(x, y), q_1, q_2)$, where $q_1 : x \to \sup(x, y)$ and $q_2 : y \to \sup(x, y)$ are the unique arrows having the given domains and codomains.

Let $z \in L$ and let $x \to z$ and $y \to z$ be arrows. This means that $x \leqslant z$ and $y \leqslant z$. We look for a unique arrow $u : \sup(x, y) \to z$ such that the following diagram is commutative:

$$\begin{tikzcd} & z & \\ x \arrow[ur] \arrow[r] & \sup(x, y) \arrow[u, dashed, "u"] & y \arrow[ul] \arrow[l] \end{tikzcd}$$

This means that $\sup(x, y) \leqslant z$. But this is true, because z is an upper bound of x and y, while $\sup(x, y)$ is the smallest upper bound of x and y.

Note that if a poset $(L, \leqslant)$ is not a lattice, two elements of L might not have a coproduct.

The construction of a coproduct may be easily generalized to an arbitrary family of elements, when $(L, \leqslant)$ is a complete lattice, that is, every family of elements of L has an infimum and a supremum.

3.2.4 Categories with Coproducts

Definition 3.3. A category $\mathcal{C}$ is said to **have (binary) coproducts** if any family of (two) objects of $\mathcal{C}$ has a coproduct.

Example 3.1. We have seen that **Set** and **Ab** have (binary) coproducts, while poset categories may not have coproducts.

Let $\mathcal{C}$ be a category with binary coproducts, and let $f : A \to B$ and $f' : A' \to B'$ be arrows in $\mathcal{C}$. Dually to the case of products, one may define a coproduct $f \oplus f' : A \oplus A' \to B \oplus B'$ of f and f'.

Consider the coproducts $(A \oplus A', p_1, p_2)$ and $(B \oplus B', q_1, q_2)$. Let $f_1 = q_1 \circ f$ and $f_2 = q_2 \circ f'$. By universal mapping property of the product $(A \oplus A', p_1, p_2)$ there is a unique arrow $u : A \oplus A' \to B \oplus B'$ such that the following diagram is commutative:

$$\begin{tikzcd}
B \arrow[r, "q_1"] & B \oplus B' & B' \arrow[l, "q_2"'] \\
A \arrow[u, "f"] \arrow[r, "p_1"'] & A \oplus A' \arrow[u, dashed, "u"'] & A' \arrow[l, "p_2"] \arrow[u, "f'"']
\end{tikzcd}$$

that is, $u \circ p_1 = f_1 = q_1 \circ f$ and $u \circ p_2 = f_2 = q_2 \circ f'$. We define $f \oplus f' = u : A \oplus A' \to B \oplus B'$.

One may prove the following result.

Proposition 3.4. Let $\mathcal{C}$ be a category with binary coproducts. Then we have a covariant functor $L : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ defined by

$$L(C, C') = C \oplus C'$$

for every object (C, C') of $\mathcal{C} \times \mathcal{C}$, and

$$L(f, f') = f \oplus f' : A \oplus A' \to B \oplus B'$$

for every arrow (f, f') from $\mathcal{C} \times \mathcal{C}$ with $f : A \to B$ and $f' : A' \to B'$.

For a category $\mathcal{C}$ with coproducts, one may generalize this construction to any finite family of arrows, and define a corresponding functor.

One may show the following associativity property by using universal mapping property of the coproduct.

Proposition 3.5.

Proposition 3.2.9 In any category $\mathcal{C}$ with binary coproducts, we have

$$A \oplus (B \oplus C) \cong (A \oplus B) \oplus C$$

where A, B, C are objects of $\mathcal{C}$.

One may show the following property.

Proposition 3.6. Let $\mathcal{C}$ be a locally small category with binary coproducts. Then for every object A of $\mathcal{C}$, the contravariant functor $\mathrm{Hom}_{\mathcal{C}}(-, A) : \mathcal{C} \to \mathbf{Set}$ preserves binary coproducts, that is, for every $B, C \in \mathcal{C}$, there is a bijection (i.e., isomorphism in **Set**):

$$\mathrm{Hom}_{\mathcal{C}}(B \oplus C, A) \cong \mathrm{Hom}_{\mathcal{C}}(B, A) \oplus \mathrm{Hom}_{\mathcal{C}}(C, A)$$

3.3 Equalizers

Definition 3.4. Let $\mathcal{C}$ be a category and let $f, g : A \to B$ be arrows in $\mathcal{C}$. An **equalizer** of f and g consists of a pair (E, e), where E is an object of $\mathcal{C}$ and $e : E \to A$ is an arrow in $\mathcal{C}$ such that $f \circ e = g \circ e$ and it has the following universal mapping property: given any object Z and any arrow $z : Z \to A$ in $\mathcal{C}$ such that $f \circ z = g \circ z$, there is a unique arrow $u : Z \to E$ in $\mathcal{C}$ such that $e \circ u = z$.

$$\begin{tikzcd}
E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
Z \arrow[u, dashed, "u"] \arrow[ur, "z"'] & &
\end{tikzcd}$$

By universal mapping property of an equalizer one deduces the following result.

Proposition 3.7. An equalizer is uniquely determined up to an isomorphism.

Proposition 3.8. Any equalizer is a monomorphism.

Proof. Let $f, g : A \to B$ be arrows in a category $\mathcal{C}$ having an equalizer (E, e). Let $\alpha, \beta : Z \to E$ be arrows in $\mathcal{C}$ such that $e \circ \alpha = e \circ \beta$. Denote $z = e \circ \alpha = e \circ \beta : Z \to A$. Hence we have

$$f \circ z = f \circ e \circ \alpha = g \circ e \circ \alpha = g \circ e \circ \beta = g \circ z.$$

By universal mapping property of the equalizer, there is a unique arrow $u : Z \to E$ such that $e \circ u = z$. But we also have $e \circ \alpha = z$ and $e \circ \beta = z$. Hence we must have $u = \alpha = \beta$. This shows that e is a monomorphism. $\square$

3.3.1 The Category Set

The equalizer of two functions $f, g : A \to B$ in **Set** is the pair (E, e), where

$$E = \{a \in A \mid f(a) = g(a)\}$$

and $e : E \to A$ is the inclusion function.

For every $a \in E$ we have

$$(f \circ e)(a) = f(e(a)) = f(a) = g(a) = g(e(a)) = (g \circ e)(a),$$

hence $f \circ e = g \circ e$.

Now let $z : Z \to A$ be a function such that $f \circ z = g \circ z$. We look for a function $u : Z \to E$ such that $e \circ u = z$. This equality is equivalent to $e(u(x)) = z(x)$ for every $x \in Z$, that is, $u(x) = z(x)$ for every $x \in Z$. Note that $z(x) \in E$, because $f(z(x)) = g(z(x))$. Also, the equality $u(x) = z(x)$ uniquely determines u. Hence (E, e) is an equalizer of f, g.

3.3.2 The Category $\mathbf{Vect}(K)$

The equalizer of two K-linear maps $f, g : A \to B$ in $\mathbf{Vect}(K)$ is the pair (E, e), where

$$E = \{a \in A \mid f(a) = g(a)\}$$

and $e : E \to A$ is the inclusion K-linear map.

As in the category **Set**, we have $f \circ e = g \circ e$. Also, for every K-linear map $z : Z \to A$ such that $f \circ z = g \circ z$, there is a unique function $u : Z \to E$ such that $e \circ u = z$. This equality is equivalent to $e(u(x)) = z(x)$ for every $x \in Z$, that is, $u(x) = z(x)$ for every $x \in Z$.

Let us show that u is a K-linear map. Let $k_1, k_2 \in K$ and $x_1, x_2 \in Z$. Then we have

$$u(k_1x_1 + k_2x_2) = z(k_1x_1 + k_2x_2) = k_1z(x_1) + k_2z(x_2) = k_1u(x_1) + k_2u(x_2)$$

Hence u is a K-linear map.

In this category the equalizer of two K-linear maps is in fact a kernel of some K-linear map, namely

$$E = \{a \in A \mid (f - g)(a) = 0\} = \mathrm{Ker}(f - g)$$

On the other hand, the kernel of a K-linear map is the equalizer of some K-linear maps, namely

$$\mathrm{Ker}(f) = \{a \in A \mid f(a) = 0\}$$

is the equalizer of the K-linear map $f : A \to B$ and the zero K-linear map $0 : A \to B$.

3.3.3 Monoid Categories

Let $(M, \cdot)$ be a monoid. Recall that it may be viewed as a monoid category, where the single object is M, the arrows are the elements of M and the composition is the multiplication of the elements of M. An equalizer of two elements $a, b \in M$ is an element $x \in M$ with $ax = bx$ and for every $z \in M$ such that $az = bz$ there is $u \in M$ such that $xu = z$. If $(M, \cdot)$ is a non-trivial group, then $a, b \in M$ with $a \neq b$ do not have an equalizer, because there is no $x \in M$ such that $ax = bx$.

Definition 3.5. We say that a category **has equalizers** if every arrows $f, g : A \to B$ have an equalizer.

Example 3.2. The categories **Set** and $\mathbf{Vect}(K)$ have equalizers, but monoid categories may not have equalizers.

3.4 Coequalizer

Definition 3.6. Let $\mathcal{C}$ be a category and let $f, g : A \to B$ be arrows in $\mathcal{C}$. A coequalizer of f and g consists of a pair (Q, q), where Q is an object of $\mathcal{C}$ and $q : B \to Q$ is an arrow in $\mathcal{C}$ such that $q \circ f = q \circ g$ and it has the following universal mapping property: given any object Z and any arrow $z : B \to Z$ in $\mathcal{C}$ such that $z \circ f = z \circ g$, there is a unique arrow $u : Q \to Z$ in $\mathcal{C}$ such that $u \circ q = z$.

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "q"] \arrow[dr, "z"'] & Q \arrow[d, dashed, "u"] \\ & & Z \end{tikzcd}$$

The following two results are dual to the corresponding ones for equalizers.

Proposition 3.9. A coequalizer is uniquely determined up to an isomorphism.

Proposition 3.10. Any coequalizer is an epimorphism.

3.4.1 The Category Set

Let $f : A \to B$ be function. Then the homogeneous relation $\ker(f) = (A, A, \mathrm{Ker}(f))$ with the graph

$$\mathrm{Ker}(f) = \{(x_1, x_2) \in A \times A \mid f(x_1) = f(x_2)\}$$

is called the **kernel** of f. Note that the kernel of f is an equivalence relation on A.

The following theorem will be useful.

Theorem 3.11 (Factorization theorem by a surjection)**.** Let $f : A \to B$ be a function and let $g : A \to C$ be a surjective function such that $\mathrm{Ker}(g) \subseteq \mathrm{Ker}(f)$. Then there is a unique function $h : C \to B$ such that $f = h \circ g$.

Let $f, g : A \to B$ be functions. We define a relation $r = (B, B, R)$ by

$$(b_1, b_2) \in R \iff \exists a \in A : b_1 = f(a) \text{ and } b_2 = g(a).$$

Let $\overline{r} = (B, B, \overline{R})$ be the smallest equivalence relation on B containing R. Then we may consider the partition

$$B/\overline{R} = \{\overline{R}\langle b\rangle \mid b \in B\}$$

of B. Consider the function $\pi : B \to B/\overline{R}$ defined by $\pi(b) = \overline{R}\langle b\rangle$. Note that the kernel of the function π is

$$\begin{aligned} \mathrm{Ker}(\pi) &= \{(b_1, b_2) \in B \times B \mid \pi(b_1) = \pi(b_2)\} \\ &= \left\{(b_1, b_2) \in B \times B \mid \overline{R}\langle b_1\rangle = \overline{R}\langle b_2\rangle\right\} = \overline{R}. \end{aligned}$$

We show that $(B/\overline{R}, \pi)$ is a coequalizer of f and g.

For every $a \in A$, we have $(f(a), g(a)) \in \overline{R}$, which implies that $\overline{R}\langle f(a)\rangle = \overline{R}\langle g(a)\rangle$. It follows that

$$(\pi \circ f)(a) = \overline{R}\langle f(a)\rangle = \overline{R}\langle g(a)\rangle = (\pi \circ g)(a),$$

hence $\pi \circ f = \pi \circ g$.

Now let $z : B \to Z$ be a function such that $z \circ f = z \circ g$. We show that

$$\overline{R} \subseteq \mathrm{Ker}(z) = \{(b_1, b_2) \in B \times B \mid z(b_1) = z(b_2)\}$$

(the kernel of the function z). Let $(b_1, b_2) \in \overline{R}$. Then there is a finite number of elements $c_0, \ldots, c_n \in B$ such that $b_1 = c_0, b_2 = c_n$ and for every $i \in \{1, \ldots, n\}$, we have either $(c_{i-1}, c_i) \in R$ or $(c_i, c_{i-1}) \in R$. We may assume that $(b_1, b_2) \in R$, because otherwise we may proceed inductively. Since $(b_1, b_2) \in R$, there is $a \in A$ such that $b_1 = f(a)$ and $b_2 = g(a)$. Then we have

$$z(b_1) = z(f(a)) = z(g(a)) = z(b_2),$$

hence $(b_1, b_2) \in \operatorname{Ker}(z)$. Thus, we have $\operatorname{Ker}(\pi) = \overline{R} \subseteq \operatorname{Ker}(z)$. Using the factorization theorem of the function z by the surjective function π, there is a unique function $u : B/\overline{R} \to Z$ such that $u \circ \pi = z$.

3.4.2 The Category Ab

The following theorem will be useful.

Theorem 3.12 (Factorization Theorem by an Epimorphism)**.** Let $f : A \to B$ be a homomorphism of abelian groups, and let $g : A \to A'$ be an epimorphism of abelian groups with $\operatorname{Ker}(g) \subseteq \operatorname{Ker}(f)$. Then there exists a unique homomorphism of abelian groups $h : A' \to B$ such that $f = h \circ g$, that is, the following diagram is commutative:

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[dr, "f"'] & A' \arrow[d, dashed, "h"] \\ & B \end{tikzcd}$$

Proof. Let $a' \in A'$. Since g is an epimorphism, there exists $a \in A$ such that $g(a) = a'$. If there exists $a_0 \in A$ such that $g(a_0) = a'$, then we have $g(a) = g(a_0)$. It follows that $g(a - a_0) = 0$, hence $a - a_0 \in \operatorname{Ker}(g) \subseteq \operatorname{Ker}(f)$. Then $f(a - a_0) = 0$, hence $f(a) = f(a_0)$.

It follows that we can define the function $h : A' \to B$ by $h(a') = f(a)$, where $f(a)$ is uniquely determined as above. We have $h(g(a)) = a$ for every $a \in A$. Hence $f = h \circ g$.

We show that h is a homomorphism of abelian groups. Let $a_1', a_2' \in A'$. Then there exist $a_1, a_2 \in A$ such that $g(a_1) = a_1'$ and $g(a_2) = a_2'$. Hence $h(a_1') = f(a_1)$ and $h(a_2') = f(a_2)$. We have

$$g(a_1 + a_2) = g(a_1) + g(a_2) = a_1' + a_2'.$$

It follows that

$$h(a_1' + a_2') = h(g(a_1 + a_2)) = f(a_1 + a_2) = f(a_1) + f(a_2) = h(a_1') + h(a_2')$$

Thus h is a homomorphism of abelian groups.

For uniqueness, suppose that there exists a homomorphism $h' : A' \to B$ such that $f = h' \circ g$. It follows that $h \circ g = h' \circ g$. Since g is an epimorphism, we have $h = h'$. $\square$

Let $(A, +)$ and $(B, +)$ be abelian groups and let $f, g : A \to B$ be group homomorphisms. Since $I = \operatorname{Im}(f - g)$ is a subgroup of B, we may consider the factor group

$$Q = B/I = \{b + I \mid b \in B\}$$

where the operation is defined by

$$(b_1 + I) + (b_2 + I) = (b_1 + b_2) + I, \quad \forall b_1, b_2 \in B$$

The factor group $B/\operatorname{Im}(f-g)$ is also called a cokernel of the group homomorphism $f-g$, and we denote it by $\operatorname{Coker}(f-g)$.

Consider the group homomorphism $q : B \to Q = B/I$ defined by $q(b) = b+I$. We have $\operatorname{Ker}(q) = \{b \in B \mid q(b) = I\} = \{b \in B \mid b+I = I\} = I$.

We show that (Q, q) is a coequalizer of f and g.

For every $a \in A$ we have $f(a) - g(a) = (f-g)(a) \in I$, hence $f(a) + I = g(a) + I$. Then for every $a \in A$ we have

$$(q \circ f)(a) = f(a) + I = g(a) + I = (q \circ g)(a),$$

hence $q \circ f = q \circ g$.

Now let $(Z, +)$ be an abelian group and let $z : B \to Z$ be a group homomorphism such that $z \circ f = z \circ g$. We show that $I \subseteq \operatorname{Ker}(z)$. To this end, let $b \in I$. Then $b = (f-g)(a)$ for some $a \in A$. It follows that

$$z(b) = z((f-g)(a)) = z(f(a) - g(a)) = z(f(a)) - z(g(a)) = 0,$$

hence $b \in \operatorname{Ker}(z)$. Thus we have $\operatorname{Ker}(q) = I \subseteq \operatorname{Ker}(z)$. Using the factorization theorem for the group homomorphism z by the epimorphism q, there is a unique group homomorphism $u : B/I \to Z$ such that $u \circ q = z$.

In this category the coequalizer of two group homomorphisms is in fact the cokernel of some group homomorphism, namely

$$Q = \operatorname{Coker}(f-g)$$

On the other hand, the cokernel of a group homomorphism is the coequalizer of some group homomorphism, namely

$$\operatorname{Coker}(f) = B/\operatorname{Im}(f) = B/\operatorname{Im}(f-0)$$

is the coequalizer of the group homomorphism $f : A \to B$ and the zero group homomorphism $0 : A \to B$.

3.4.3 Monoid Categories

Let $(M, \cdot)$ be a monoid, which may be viewed as a monoid category. A coequalizer of two elements $a, b \in M$ is an element $x \in M$ with $xa = xb$ and for every $z \in M$ such that $za = zb$ there is $u \in M$ such that $ux = z$. If $(M, \cdot)$ is a non-trivial group, then $a, b \in M$ with $a \neq b$ do not have a coequalizer, because there is no $x \in M$ such that $xa = xb$.

Definition 3.7. We say that a category **has coequalizers** if every arrows $f, g : A \to B$ have a coequalizer.

Example 3.3. The categories **Set** and **Ab** have coequalizers, while monoid categories may not have coequalizers.

Chapter 4

Limits and Colimits

Chapter 5

Naturality

Chapter 6

Categories of Diagrams

Chapter 7

Adjoints

Chapter 8

Additive Categories

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